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A
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OF
ARITHMETICAL AND ALGEBRAIC
PROBLEMS AND FORMULÆ;

BY

MEIER HIRSCH:

TRANSLATED

FROM THE ORIGINAL GERMAN,

AND

ADAPTED TO THE USE OF THE AMERICAN STUDENT.

BY FRANCIS J. GRUND,
Teacher of Mathematics.

BOSTON:
CARTER, HENDEE AND BABCOCK.
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TO

THE HONORABLE

NATHANIEL BOWDITCH, LL. D., F.R.S.,

President of the American Academy of Arts and Sciences,

This Collection

OF

PROBLEMS AND FORMULÆ

IS

RESPECTFULLY DEDICATED,

BY

THE TRANSLATOR.

PREFACE.

IN offering to the public a translation of Meier Hirsch's problems and formulæ, it is needless to say anything as to their merit or usefulness. The author, *Meier Hirsch*, is well known as one of the ablest mathematicians in Europe, and, perhaps, as the best teacher of our time. His problems have run through a great number of editions in Germany, and have been translated into most European languages. A translation of them has been published in England, by Rev. S. A. Ross; but the difficulty of procuring the English copy, its high price, and its great incorrectness, probably arising from the translator's imperfect knowledge of the German language, were sufficient inducements for me to attempt an American translation of the original, which might be exempt from some of the gross errors which occur in the English copy I can claim no merit beyond that of endeavouring to give the true sense of the original. No essential alteration has been made, except in the coins, weights, and measures, where, instead of the German, I have substituted those in general use in the United States.

A few arithmetical problems, however, have been added, which, it is hoped, will not be considered as an incumbrance to the work. Some of the promiscuous problems on Fractions, (pages 21 and 22) are taken from the collection of Dr Lehmus, of Berlin, and belong to the only class of reductions which Meier Hirsch seems to have overlooked in his own.

FRANCIS J. GRUND.

Boston, June 5th, 1831.

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ARITHMETIC AND ALGEBRAIC PROBLEMS AND FORMULÆ.

SECTION I.

CONTAINING

EXAMPLES IN ARITHMETIC.

a. TO FIND THE PRIME FACTORS OF COMPOUND NUMBERS.

Questions to be answered. What are prime numbers? What compound numbers? How do you find the factors of a compound number?

EXAMPLES.

1) $450 = 2 . 3 . 3 . 5 . 5$

2) $1260 = 2 . 2 . 3 . 3 . 5 . 7$

3) $4590 = 2 . 3 . 3 . 3 . 5 . 17$

4) $14500 = 2 . 2 . 5 . 5 . 5 . 29$

5) $7835 = 5 . 1567$

6) $989 = 23 . 43$

7) $20685 = 3 . 5 . 7 . 197$

8) $3366 = 2 . 3 . 3 . 11 . 17$

9) $1089 = 3 . 3 . 11 . 11$

10) 729 = 3 . 3 . 3 . 3 . 3 . 3

11) 256 = 2 . 2 . 2 . 2 . 2 . 2 . 2 . 2

12) 625 = 5 × 5 × 5

b. TO FIND THE SMALLEST COMMON MULTIPLE OF SEVERAL GIVEN NUMBERS.

Questions to be answered. What is meant by a multiple of several given numbers? What is meant by the *smallest* common multiple of several given numbers? How do you find it?*

EXAMPLES.

1) given 2, 4, 8
Smallest multiple = 8

2) given 2, 3, 6, 5
Smallest multiple = 30

3) given 2, 3, 4, 5, 6, 10
Smallest multiple = 60

4) given 2, 3, 4, 5, 6, 10, 12, 15
Smallest multiple = 60

5) given 4, 5, 7, 6, 3, 2, 10, 14
Smallest multiple = 420

6) given 1, 2, 3, 4, 5, 6, 7, 8, 9, 10
Smallest multiple = 2520

7) given 10, 20, 30, 40, 50, 60, 70, 80, 90, 100
Smallest multiple = 25200

---

* The smallest common multiple of several given numbers is the product of all their prime factors.

8) given 3, 6, 9, 12, 15, 18
Smallest multiple = 180

9) given 2, 4, 8, 16, 32, 64, 128
Smallest multiple = 128

10) given 2, 3, 4, 7, 8, 16, 20, 25, 37, 9
Smallest multiple = 25200

---

c. ADDITION OF VULGAR FRACTIONS.

Questions to be answered. What is a fraction? What a proper fraction? What an improper fraction? What is the numerator of a fraction? What its denominator? Where does the numerator stand? Where the denominator? How are fractions added? What is meant by the common denominator of several fractions? How are fractions reduced to a common denominator? How are fractions subtracted? How multiplied? How divided?

EXAMPLES.

1) $\frac{3}{4}$	2) $4\frac{1}{2}$	3) $9\frac{5}{7}$	4) $63\frac{1}{2}$
$\frac{1}{2}$	$14\frac{3}{4}$	$14\frac{3}{5}$	$14\frac{3}{4}$
$\frac{4}{5}$	$29\frac{5}{8}$	$12\frac{11}{12}$	$15\frac{7}{8}$
$\frac{9}{10}$	$46\frac{1}{3}$	$9\frac{1}{3}$	$19\frac{11}{16}$
$\frac{11}{12}$	$45\frac{5}{6}$	$45\frac{3}{4}$	$24\frac{11}{12}$
$\frac{13}{15}$	$19\frac{11}{12}$	$15\frac{1}{8}$	$\frac{7}{8}$
$4\frac{11}{15}$	$160\frac{23}{24}$	$107\frac{123}{280}$	$139\frac{29}{48}$

5) $\frac{4}{5} + \frac{1}{2} + \frac{3}{4} + \frac{7}{8} + \frac{1}{3} = \frac{391}{120} = 3\frac{31}{120}$

6) $\frac{3}{4} + \frac{7}{8} + 5\frac{1}{2} + 7\frac{4}{5} + 4 = 16 + \frac{117}{40} = 18\frac{37}{40}$

7 $\frac{6}{7} + \frac{3}{4} + \frac{5}{7} + \frac{1}{3} + \frac{1}{4} = \frac{244}{84} = 2\frac{76}{84} = 2\frac{19}{21}$

8) $9\frac{3}{4} + 7\frac{2}{3} + 5\frac{7}{8} + 11\frac{5}{6} = 32 + \frac{75}{24} = 35\frac{1}{8}$

9) $7\frac{1}{2} + 11\frac{3}{4} + 25\frac{7}{9} + 9\frac{3}{5} = 52 + \frac{473}{180} = 54\frac{113}{180}$

10) $4\frac{3}{4} + 7\frac{1}{3} + 4\frac{7}{9} + \frac{1}{3} + 1 = 16 + \frac{79}{36} = 18\frac{7}{36}$

11) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = \frac{31}{42}$

12) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} = 1 + \frac{81}{243} + \frac{27}{243}$
$+ \frac{9}{243} + \frac{3}{243} + \frac{1}{243} = 1\frac{121}{243}$

---

d. SUBTRACTION OF VULGAR FRACTIONS.

EXAMPLES.

$$1)\ \begin{array}{r}\frac{1}{2}\\ \frac{1}{4}\\ \hline \frac{1}{4}\end{array} \qquad 2)\ \begin{array}{r}\frac{2}{3}\\ \frac{1}{4}\\ \hline \frac{5}{12}\end{array} \qquad 3)\ \begin{array}{r}\frac{5}{6}\\ \frac{3}{4}\\ \hline \frac{1}{12}\end{array} \qquad 4)\ \begin{array}{r}\frac{7}{8}\\ \frac{3}{4}\\ \hline \frac{1}{8}\end{array} \qquad 5)\ \begin{array}{r}\frac{11}{12}\\ \frac{9}{10}\\ \hline \frac{1}{60}\end{array}$$

$$6)\ \begin{array}{r}10\frac{1}{2}\\ 9\frac{3}{4}\\ \hline \frac{3}{4}\end{array} \qquad 7)\ \begin{array}{r}12\frac{1}{3}\\ 9\frac{1}{2}\\ \hline 2\frac{5}{6}\end{array} \qquad 8)\ \begin{array}{r}15\frac{4}{5}\\ 9\frac{1}{3}\\ \hline 6\frac{7}{15}\end{array} \qquad 9)\ \begin{array}{r}124\frac{5}{6}\\ 9\frac{7}{8}\\ \hline 114\frac{23}{24}\end{array}$$

10) $\frac{3}{4} - \frac{1}{2} = \frac{1}{4}$ 11) $7\frac{4}{5} - \frac{7}{8} = 6\frac{37}{40}$

12) $15 - \frac{3}{7} = 14\frac{4}{7}$ 13) $1 - \frac{11}{12} = \frac{1}{12}$

14) $27\frac{1}{2} - 3\frac{1}{2} - 4\frac{3}{4} - 7\frac{4}{5} - \frac{7}{8} = 27\frac{1}{2} - 16\frac{37}{40} = 10\frac{23}{40}$

15) $234\frac{7}{8} - 5\frac{3}{4} - 3\frac{1}{2} - 4\frac{3}{5} - 117\frac{5}{6} - 10\frac{11}{12}$
$= 234\frac{7}{8} - 142\frac{3}{5} = 92\frac{11}{40}$

16) $14\frac{7}{18} - \frac{3}{4} - \frac{5}{9} - \frac{3}{5} - \frac{1}{3} = 14\frac{7}{18} - 2\frac{43}{180} = 12\frac{27}{180}$
$= 12\frac{3}{20}$

17) $1\frac{7}{8} - \frac{3}{4} - \frac{3}{8} - \frac{3}{16} - \frac{3}{32} = 1\frac{28}{32} - 1\frac{13}{32} = \frac{15}{32}$

18) $1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \frac{1}{16} - \frac{1}{32} = \frac{1}{32}$

e. MULTIPLICATION OF VULGAR FRACTIONS.

EXAMPLES.

1) $\frac{3}{4} \times 2 = \frac{6}{4} = \frac{3}{2} = 1\frac{1}{2}$

2) $4\frac{7}{8} \times 3 = 14\frac{5}{8}$

3) $\frac{1}{2} \times \frac{3}{4} = \frac{3}{8}$

4) $\frac{3}{4} \times \frac{4}{5} = \frac{12}{20} = \frac{3}{5}$

5) $4\frac{1}{2} \times \frac{3}{4} = 3\frac{3}{8}$

6) $7\frac{3}{4} \times 4\frac{1}{2} = 34\frac{7}{8}$

7) $14\frac{3}{4} \times 1\frac{1}{4} = 18\frac{7}{16}$

8) $42\frac{7}{8} \times 7\frac{3}{5} = 325\frac{17}{20}$

9) $25\frac{3}{4} \times 62 = 1596\frac{1}{2}$

10) $67\frac{1}{8} \times 10 = 671\frac{1}{4}$

11) $47\frac{7}{8} \times 3 = 143\frac{5}{8}$

12) $3 \times 47\frac{7}{8} = 143\frac{5}{8}$

13) $4\frac{7}{8} \times \frac{3}{4} = 3\frac{21}{32}$

14) $\frac{3}{4} \times 4\frac{7}{8} = 3\frac{21}{32}$

15) $\frac{1}{2} \times \frac{2}{3} \times \frac{4}{5} \times \frac{7}{8} \times \frac{3}{4} \times \frac{1}{7} = \frac{1}{40}$[*]

16) $\frac{3}{4} \times \frac{7}{8} \times \frac{4}{5} \times 1\frac{3}{4} \times 5 = 4\frac{19}{32}$

17) $3\frac{3}{4} \times 2\frac{1}{2} \times 1\frac{3}{5} \times 3\frac{3}{5} = \frac{15}{4} \times \frac{5}{2} \times \frac{8}{5} \times \frac{18}{5} = 54$

18) $\frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} \times \frac{1}{5} \times \frac{1}{6} = \frac{1}{720}$

19) $\frac{1}{10} \times \frac{3}{10} \times \frac{5}{10} \times \frac{7}{10} = \frac{105}{10000} = \frac{21}{2000}$

20) $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{32}$

21) $3\frac{3}{4} \times 4\frac{4}{5} \times 12 \times 3 = 648$

22) $7\frac{3}{4} \times \frac{4}{5} \times \frac{7}{8} \times 925 = 5018\frac{1}{8}$[†]

---

[*] By striking out the equal factors in numerator and denominator.

[†] The denominator 5 (of the fraction $\frac{4}{5}$) divides 925 without remainder.

f. DIVISION OF VULGAR FRACTIONS.

EXAMPLES.

1) $5 : \frac{1}{2} = 10$

2) $6 : \frac{1}{3} = 18$

3) $6 : \frac{2}{3} = 9$

4) $15 : \frac{4}{5} = \frac{75}{4} = 18\frac{3}{4}$

5) $8 : 2\frac{1}{2} = 8 : \frac{5}{2} = \frac{16}{5} = 3\frac{1}{5}$

6) $9 : 3\frac{1}{3} = 9 : \frac{10}{3} = \frac{27}{10} = 2\frac{7}{10}$

7) $25 : 9\frac{1}{2} = 25 : \frac{19}{2} = \frac{50}{19} = 2\frac{12}{19}$

8) $35 : 4\frac{4}{5} = 35 : \frac{24}{5} = \frac{175}{24} = 7\frac{7}{24}$

9) $25 : 5\frac{2}{3} = 25 : \frac{17}{3} = \frac{75}{17} = 4\frac{7}{17}$

10) $45 : 6\frac{2}{7} = 45 : \frac{44}{7} = \frac{315}{44} = 7\frac{7}{44}$

11) $3 : 4\frac{1}{5} = 3 : \frac{21}{5} = \frac{15}{21} = \frac{5}{7}$

12) $5 : 9\frac{2}{3} = 5 : \frac{29}{3} = \frac{15}{29}$

13) $1 : \frac{2}{3} = \frac{3}{2} = 1\frac{1}{2}$

14) $1 : \frac{3}{4} = \frac{4}{3} = 1\frac{1}{3}$

15) $1 : 2\frac{1}{2} = 1 : \frac{5}{2} = \frac{2}{5}$

16) $\frac{1}{2} : \frac{1}{3} = \frac{3}{2} = 1\frac{1}{2}$

17) $\frac{1}{4} : \frac{1}{5} = \frac{5}{4} = 1\frac{1}{4}$

18) $\frac{2}{3} : \frac{1}{6} = \frac{10}{3} = 3\frac{1}{3}$

19) $\frac{3}{5} : \frac{1}{4} = \frac{12}{5} = 2\frac{2}{5}$

20) $\frac{3}{4} : \frac{4}{5} = \frac{15}{16}$

21) $\frac{1}{2} : \frac{7}{8} = \frac{8}{14} = \frac{4}{7}$

22) $5\frac{3}{4} : \frac{7}{8} = \frac{23}{4} : \frac{7}{8} = \frac{23}{4} \times \frac{8}{7} = 23 \times \frac{2}{7} = \frac{46}{7} = 6\frac{4}{7}$

23) $7\frac{4}{5} : 3\frac{1}{2} = \frac{39}{5} : \frac{7}{2} = \frac{78}{35} = 2\frac{8}{35}$

24) $12\frac{3}{4} : 4\frac{3}{4} = \frac{51}{4} : \frac{19}{4} = \frac{51}{19} = 2\frac{13}{19}$

25) $15\frac{2}{5} : 3\frac{3}{5} = \frac{77}{5} : \frac{18}{5} = \frac{77}{18} = 4\frac{5}{18}$

26) $15\frac{5}{6} : 3\frac{1}{6} = \frac{95}{6} : \frac{19}{6} = \frac{95}{19} = 5$

27) $27\frac{3}{4} : 5\frac{1}{8} = \frac{111}{4} : \frac{41}{8} = \frac{111}{4} \times \frac{8}{41} = 111 \times \frac{2}{41}$*
$= \frac{222}{41} = 5\,\frac{17}{41}$

28) $31\frac{4}{5} : 7\frac{3}{4} = \frac{159}{5} : \frac{31}{4} = \frac{636}{155} = 4\frac{16}{636} = 4\frac{4}{159}$

29) $7\frac{4}{5} : 4\frac{1}{3} = \frac{39}{5} : \frac{13}{3} = \frac{117}{65} = 1\frac{52}{65}$

30) $7\frac{4}{5} : 2\frac{3}{5} = \frac{39}{5} : \frac{13}{5} = 3$

31) $1\frac{2}{3} : 5\frac{5}{7} = \frac{5}{3} : \frac{40}{7} = \frac{35}{120} = \frac{7}{24}$

32) $5\frac{3}{4} : 12\frac{4}{5} = \frac{23}{4} : \frac{64}{5} = \frac{115}{256}$

33) $17\frac{2}{3} : 25\frac{3}{4} = \frac{53}{3} : \frac{103}{4} = \frac{212}{309}$

34) $2 : \frac{2}{3} = \frac{6}{2} = 3$

35) $3 : \frac{3}{4} = \frac{12}{3} = 4$

36) $7 : \frac{4}{5} = \frac{35}{4} = 8\frac{3}{4}$

37) $6 : 1\frac{3}{4} = 6 : \frac{7}{4} = \frac{24}{7} = 3\frac{3}{7}$

38) $1 : \frac{4}{5} = \frac{5}{4} = 1\frac{1}{4}$

39) $1 : 3\frac{4}{5} = 1 : \frac{19}{5} = \frac{5}{19}$

40) $1 : 1\frac{1}{2} = 1 : \frac{3}{2} = \frac{2}{3}$

41) $6 : 2\frac{2}{3} = 6 : \frac{8}{3} = \frac{18}{8} = \frac{9}{4} = 2\frac{1}{4}$

42) $7 : 3\frac{3}{4} = 7 : \frac{15}{4} = \frac{28}{15} = 1\frac{13}{15}$

43) $64 : 20\frac{1}{4} = 64 : \frac{81}{4} = \frac{256}{81} = 3\frac{13}{81}$

44) $52 : 5\frac{1}{3} = 52 : \frac{16}{3} = \frac{156}{16} = 9\frac{12}{16} = 9\frac{3}{4}$

45) $57 : 14\frac{2}{5} = 57 : \frac{72}{5} = \frac{285}{72} = 3\frac{69}{72} = 3\frac{23}{24}$

---

* 4 into 8 go twice.

46) $35 : 2\frac{1}{2} = 35 : \frac{5}{2} = 7 : \frac{1}{2}$* $= 14$

47) $14 : 3\frac{1}{2} = 14 : \frac{7}{2} = 2 : \frac{1}{2} = 4$

48) $15 : 3\frac{3}{4} = 15 : \frac{15}{4} = 1 : \frac{1}{4} = 4$

49) $27 : \frac{1}{9} = 243$

50) $543 : \frac{4}{5} = \frac{2715}{4} = 678\frac{3}{4}$

g. FINDING OF THE GREATEST COMMON DIVISOR.

Questions to be answered. What is a common divisor of two or more numbers? What is the *greatest* common divisor of two numbers? The greatest common divisor of two or more numbers is the product of all the prime factors common to these numbers. The prime factors of one number, which are not common to the other, may be suppressed without affecting the greatest common divisor of these numbers. To what abbreviations does this principle give rise in finding the greatest common divisor of two numbers? What general rule is there for finding the greatest common divisor of two numbers? If a number divides divisor and dividend of a division, it must divide the remainder. Why? If it divides the remainder and divisor, it must divide the dividend. Why? Two numbers which have no greater common divisor than 1, are called prime with regard to each other.

EXAMPLES.

1) given 45, 12, greatest common divisor = 3

2) given 125, 75, ' ' ' = 25†

---

* Dividend and divisor divided by 5.

† 125, = 5 . 5 . 5, 75 = 5 . 5 . 3, greatest common divisor = 5 . 5 = 25.

3) given 450, 50, greatest common divisor = 50
4) given 752, 156, ‘ ‘ ‘ = 4
5) given 840, 146, ‘ ‘ ‘ = 2
6) given 4770, 4590 ‘ ‘ ‘ = 90
7) given 1530, 4590, ‘ ‘ ‘ = 1530
8) given 4500, 9600, ‘ ‘ ‘ = 300
9) given 17200, 4720, ‘ ‘ ‘ = 80
10) given 481, 703, ‘ ‘ ‘ = 37
11) given 8437, 574, ‘ ‘ ‘ = 1*
12) given 1225, 592, ‘ ‘ ‘ = 1
13) given 1961, 2025, ‘ ‘ ‘ = 1
14) given 41175, 25925, ‘ ‘ ‘ = 1525
15) given 3285, 9125, ‘ ‘ ‘ = 365
16) given 41223, 33522, ‘ ‘ ‘ = 453
17) given 104604, 87170, ‘ ‘ ‘ = 17434
18) given 888525, 98725, ‘ ‘ ‘ = 98725
19) given 45000, 4500, ‘ ‘ ‘ = 4500
20) given 36000, 360, ‘ ‘ ‘ = 360

---

*These two numbers (8437, 574) are prime with regard to each other. The same is the case with No. 12 and 13.

h. REDUCTION OF FRACTIONS TO THEIR LOWEST TERMS.

Questions to be answered. Is the value of a fraction altered when both numerator and denominator, are divided by the same number? What is the advantage of reducing a fraction to its lowest term?

EXAMPLES.

1) $\frac{4}{8} = \frac{1}{2}$
2) $\frac{8}{32} = \frac{1}{4}$
3) $\frac{20}{80} = \frac{1}{4}$
4) $\frac{25}{75} = \frac{1}{3}$
5) $\frac{36}{48} = \frac{3}{4}$
6) $\frac{140}{840} = \frac{1}{6}$
7) $\frac{50}{450} = \frac{1}{9}$
8) $\frac{4590}{4770} = \frac{51}{53}$
9) $\frac{1965}{2025} = \frac{131}{135}$
10) $\frac{3285}{9125} = \frac{9}{25}$
11) $\frac{33522}{41223} = \frac{74}{91}$
12) $\frac{39496}{69118} = \frac{4}{7}$
13) $\frac{63206}{68952} = \frac{11}{12}$
14) $\frac{27865}{44584} = \frac{5}{8}$
15) $\frac{58760}{66105} = \frac{8}{9}$
16) $\frac{4104}{4560} = \frac{9}{10}$
17) $\frac{475}{4750} = \frac{1}{10}$
18) $\frac{4500}{450000} = \frac{1}{100}$
19) $\frac{360}{14400} = \frac{1}{40}$
20) $\frac{99}{9999} = \frac{11}{1111} = \frac{1}{101}$

DECIMAL FRACTIONS.

Questions to be answered. What is meant by a Decimal Fraction? What change does it undergo when the Decimal Point is moved one place further to the right or left? How is a Vulgar Fraction converted into a Decimal Fraction? What is a Period in Decimal Fractions? How are Decimal Fractions added, subtracted, multiplied, and divided? When a Vulgar Fraction is transformed into a Decimal, how many Digits can such a Period then contain for the utmost? If it is allowed in the Products and Quotients to omit the last Decimal Digits, how can you abridge Multiplication and Division of Decimal Fractions? Wherein consists the advantage of such an abridgment, and how is it performed?

1) TRANSFORMATION OF VULGAR FRACTIONS INTO DECIMALS.

EXAMPLES.

1) $\frac{1}{2} = 0{,}5$

2) $\frac{3}{4} = 0{,}75$

3) $\frac{13}{16} = 0{,}8125$

4) $\frac{17}{20} = 0{,}85$

5) $\frac{3}{40} = 0{,}075$

6) $\frac{17}{125} = 0{,}136$

7) $\frac{7}{800} = 0{,}00875$

8) $\frac{372}{1250} = 0{,}2976$

9) $\frac{11}{16000} = 0{,}0006875$

10) $\frac{15}{1280} = 0{,}01171875$

11) $\frac{347}{25600} = 0{,}0135546875$

12) $\frac{1}{10000} = 0{,}0001$

13) $\frac{3476}{15625} = 0{,}222464$

14) $\frac{1}{2048000} = 0{,}00000048828125$

15) $\frac{2}{3} = 0{,}666666....$

16) $\frac{3}{7} = 0{,}4285714....$ (Period 428571)

17) $\frac{15}{17} = 0{,}88235....$ (Period 8823529411764705)

18) $\frac{3}{35} = 0{,}0857142857....$ (Period 857142)

19) $\frac{17}{19} = 0{,}894736842105263157....$

20) $\frac{19}{1365} = 0{,}0139194139....$

21) $\frac{706}{5907} = 0{,}1195192144....$

22) $\frac{1}{679} = 0{,}0014727540....$

23) $\frac{1}{8076934} = 0{,}0000001238....$

24) $\frac{13}{569437} = 0{,}0000228295....$

25) $\frac{130}{2769135} = 0{,}0000469460....$

26) $\frac{17}{30000} = 0{,}00056666....$

27) $\frac{47}{71000000} = 0{,}0000006619....$

2) ADDITION.

EXAMPLES.

1) 0,857
0,678
1,535

2) 1,007
2,346
3,353

3) 0,00076
13,795
13,79576

4) 12,0134
196,785
7,00006
215,79846

5) 0,90058
7,634
3,007956
11,542536

6) 7,345
8,26
37,534
19,0005
10,94
103,729
186,8085

7) 7,6
138,05934
15,4
10,76
0,3592176
1365,7
37,6483
0,005
1575,5318576

8) 19,3576
17,2340
7652,007
0,5
39,069534
7,83
5,69784
2,350006
7744,04598

9) 113,67849
76,859
9,7
5,
152,6043
7,85976
9,437
8,65
7,94
391,72855

3) SUBTRACTION.

EXAMPLES.

1) $$\begin{array}{r} 0,947 \\ 0,195 \\ \hline 0,752 \end{array}$$

2) $$\begin{array}{r} 9,567 \\ 3,078 \\ \hline 6,489 \end{array}$$

3) $$\begin{array}{r} 213,5734 \\ 87,6572 \\ \hline 125,9162 \end{array}$$

4) $$\begin{array}{r} 54,763 \\ 0,921 \\ \hline 53,842 \end{array}$$

5) $$\begin{array}{r} 73,5673 \\ 12,889 \\ \hline 60,6783 \end{array}$$

6) $$\begin{array}{r} 385,76943 \\ 72,57 \\ \hline 313,19943 \end{array}$$

7) $$\begin{array}{r} 27,003 \\ 7,6854 \\ \hline 19,3176 \end{array}$$

8) $$\begin{array}{r} 129,57 \\ 6,894356 \\ \hline 122,675644 \end{array}$$

9) $$\begin{array}{r} 0,975 \\ 0,483764 \\ \hline 0,491236 \end{array}$$

10) $$\begin{array}{r} 23,005 \\ 4,76943 \\ \hline 18,23557 \end{array}$$

11) $$\begin{array}{r} 96,5 \\ 0,000783 \\ \hline 96,499217 \end{array}$$

12) $$\begin{array}{r} 0,5 \\ 0,0003 \\ \hline 0,4997 \end{array}$$

---

4) MULTIPLICATION.

EXAMPLES.

1) $3,57 \times 6 = 21,42$

2) $5,798 \times 18 = 104,364$

3) $0,5 \times 36000 = 18000$

4) $0,00563 \times 17 = 0,09571$

5) $0,0000054 \times 3785 = 0,020439$

6) $3,7 \times 2,6 = 9,62$

7) $5,78 \times 3,4 = 19,652$

8) $3,9765 \times 4,378 = 17,409117$

9) $32,76859 \times 13,0076 = 426,240711284$

10) $138,5 \times 7,695708 = 1065,855558$

11) $0,43 \times 0,65 = 0,2795$

12) $0,576 \times 0,3854 = 0,2219904$

13) $0,005 \times 0,017 = 0,000085$

14) $0,007853 \times 0,00476 = 0,00003738028$

15) $113,5 \times 0,072 = 8,172$

16) $0,372106 \times 0,0054 = 0,0020093724$

17) $0,137 \times 0,00056 = 0,00007672$

18) $0,376 \times 0,0076894 = 0,0028912144$

ABRIDGED MULTIPLICATION.

1)
$$\begin{array}{r} 7{,}65340958 \\ \times\quad 2{,}56307 \\ \hline 1530681916 \\ 382670479 \\ 45920457 \\ 2296022 \\ 53573 \\ \hline 19{,}61622447 \end{array}$$

2)
$$\begin{array}{r} 0{,}7653478 \\ \times\quad 0{,}3576 \\ \hline 22960434 \\ 3826739 \\ 535742 \\ 45920 \\ \hline 0{,}27368835 \end{array}$$

3)
$$\begin{array}{r} 8{,}56794323 \\ \times\quad 0{,}5284765 \\ \hline 4283971615 \\ 171358864 \\ 68543545 \\ 3427177 \\ 599755 \\ 51407 \\ 4283 \\ \hline 4{,}527956646 \end{array}$$

4)
$$\begin{array}{r} 37{,}346859416 \\ \times\quad 0{,}007003458 \\ \hline 261428015912 \\ 112040578 \\ 14938743 \\ 1867342 \\ 298774 \\ \hline 0{,}261557161349 \end{array}$$

5)
$$\begin{array}{r} 0{,}076934210834 \\ \times\quad 0{,}000003057026 \\ \hline 230802632502 \\ 3846710541 \\ 538539475 \\ 1538684 \\ 461605 \\ \hline 0{,}000000235289882807 \end{array}$$

6)
$$\begin{array}{r} 15{,}7356783 \\ \times\quad 2{,}564725 \\ \hline 314713566 \\ 78678391 \\ 9441406 \\ 629426 \\ 141620 \\ 3147 \\ 786 \\ \hline 40{,}3608342 \end{array}$$

5) DIVISION.

EXAMPLES.

1) 5,64 : 2 = 2,82

2) 7,5832 : 8 = 0,9479

3) 0,357642 : 6 = 0,059607

4) 0,32769414 : 18 = 0,01820523

5) 0,01765125 : 375 = 0,00004707

6) 75 : 16 = 4,6875

7) 731 : 8 = 91,375

8) 3,54 : 7 = 0,50571428.... (Period 571428)

9) 8,2356 : 17 = 0,484447.... (Period 4705882352941176)

10) 273,694 : 543 = 0,50404051...

11) 6938,57 : 276 = 25,13974637....

12) 0,000215 : 316 = 0,0000006803797...

13) 400 : 0,25 = 1600

14) 378 : 0,01 = 37800

15) 5640 : 0,0015 = 3760000

16) 183260 : 0,476 = 385000

17) 1 : 0,24 = 4,16666...

18) 2,53944 : 7,2 = 0,3527

19) 0,02382245 : 0,37 = 0,064385

20) 1114,869145005 : 0,385 = 2895,764013

21) 56,4 : 0,00015 = 376000

22) 10287,36 : 0,0036 = 2857600

23) 0,0001 : 0,02 = 0,005

24) 145,817 : 0,0563 = 2590

25) 374 : 2,4 = 155,833333...

26) 15,713 : 18,13 = 0,86668505...

27) 137,51634 : 27,65 = 4,97346618...

28) 0,5 : 76,91342 = 0,00650081...

29) 0,046 : 0,00762089 = 6,0360404...

30) 1 : 3,2561047 = 0,30711543...

31) 38076 : 137 = 277,92700729...

ABRIDGED DIVISION.

1)

```
         |3,716048
 7,632035|2,053804...
 7 432096|
 --------
   199939
   185802
   ------
    14137
    11148
    -----
     2989
     2972
     ----
       17
       14
       --
        3
```

2)

```
          |15,314865
 2,0000000|0,13059209...
 1 5314865|
 ---------
   4685135
   4594459
   -------
     90676
     76574
     -----
     14102
     13782
     -----
       320
       306
       ---
        14
        13
        --
         1
```

3)

```
          |0,3547808
 10,926954|30,79917...
 10 643424|
 ---------
    283530
    248346
    ------
     35184
     31930
     -----
      3254
      3192
      ----
        62
        35
        --
        27
        24
        --
         3
```

4)

```
           |0,0035843297
 3,00000000|836,97659...
 2 86746376|
 ----------
   13253624
   10752989
   --------
    2500635
    2150597
    -------
     350038
     322588
     ------
      27450
      25090
      -----
       2360
       2150
       ----
        210
        179
        ---
         31
         31
         --
```

6) PROMISCUOUS EXAMPLES IN FRACTIONS.*

EXAMPLES.

1) $\frac{5}{42} + \frac{3}{154} - \frac{2}{231} = \frac{10}{77}$

2) $(\frac{7}{8} + \frac{1}{3}) \times \frac{3}{4} = \frac{29}{24} \times \frac{3}{4} = \frac{29}{8} \times \frac{1}{4} = \frac{29}{32}$

3) $(\frac{4}{5} - \frac{1}{2} + \frac{2}{3}) \times \frac{4}{5} = \frac{29}{30} \times \frac{4}{5} = \frac{116}{150} = \frac{58}{75}$

4) $(\frac{3}{4} + \frac{1}{2} - \frac{5}{6}) : (\frac{4}{5} - \frac{1}{3}) = \frac{5}{12} : \frac{7}{15} = \frac{25}{28}$

5) $(\frac{3}{35} + \frac{11}{14}) : \frac{2}{21} - \frac{3}{4} = \frac{61}{70} : \frac{2}{21} - \frac{3}{4} = \frac{183}{20} - \frac{3}{4}$
$= \frac{183}{20} - \frac{15}{20} = 8\frac{2}{5}$

6) $(\frac{7}{8} + 0,453) : 0,46 = 1,328 : 0,46 = 2,886956....$

7) $(\frac{3}{4} - 0,475 + \frac{3}{5}) \times 6,753 = 0,875 \times 6,753$
$= 5,908875$

8) $(3\frac{2}{3} + 0,4985 - 2,752) : 6,75 = 1,413166.... : 6,75$
$= 0,20935....$

9) $(5\frac{4}{5} + 7,35) : (4,375 - \frac{1}{2}) = 13,15 : 3,875$
$= 3,39354....$

10) $0,7 : (2,13 - 0,5) + 1,25 = 0,429447... + 1,25$
$= 1,679447....$

---

* These may be solved by the pupils when they have studied Algebra for some time, and become familiar with the use of signs, brackets, &c.

11) $\dfrac{4,37 - 1,75}{4\frac{3}{4} - 2\frac{1}{2}} = \dfrac{2,62}{2,25} = 1,16444....$

12) $\left(\dfrac{0,12}{2,75} + 1,314\right) : 2 = 1,3576363... : 2 = 0,67881818...$

13) $\dfrac{6 + 0,34 : 2 - 5}{(6 + 0,34) : 2 - 3} = \dfrac{1,17}{0,17} = 6,882352....$

14) $\dfrac{5\frac{1}{2} + \dfrac{7\frac{1}{4}}{0,5} - 0,725}{(4 + 3,45) : 2\frac{1}{2}} = \dfrac{19,275}{2,98} = 6,46812...$

15) $\dfrac{12\frac{1}{2} + 2\frac{1}{4}}{45} : \dfrac{9\frac{1}{3} - 4\frac{1}{5}}{0,75} = 0,3277.. : 6,8444...$

$= 0,047878....$

16) $\dfrac{9\frac{4}{5} - \frac{1}{2}}{3\frac{3}{4}} \times \dfrac{4\frac{1}{5} - 3\frac{1}{3}}{2\frac{1}{2}} = \dfrac{\frac{93}{10}}{\frac{23}{4}} \times \dfrac{\frac{13}{15}}{\frac{5}{2}} = \frac{372}{230} \times \frac{26}{75}$

$= \frac{9672}{17250} = \frac{1612}{2875}$

17) $[(\frac{3}{35} + \frac{1}{14}) : \frac{2}{21} - \frac{3}{4}] \times \frac{2}{3} - \frac{1}{10} = \frac{9}{10} \times \frac{2}{3} - \frac{1}{10}$

$= \frac{3}{5} - \frac{1}{10} = \frac{1}{2}$

18) $[0,7 : (2,13 - 0,7) + 1,03] \times 0,103 = 0,1565....$

19) $\left(\dfrac{0,12}{3,4} + 1,314\right) : 2 + 0,3 = 0,974647...$

20) $\dfrac{3 + 0,34 : 2 - 1,2}{(3 + 0,34) : 2 - 1,2} : \dfrac{3 + 0,34 : (2 - 1,2)}{(3 + 0,34) : (2 - 1,2)} = 5,109...$

1) What part of a year is a day, calculating the year at 365 days and 6 hours? Ans. 0·0027378507....
2) But what part is it, when the year, as is right, is computed to average 365,2422453 days? Ans. 0,0027379034....
3) We divide the circumference of a circle into 360 degrees, the degree into 60 minutes, the minute into 60 seconds; the French, however, divide it into 400 degrees, and express the smallest portions by decimals of the degree. How much then do our degree, minute, and second amount to in French degrees? Ans. 1° = 1°, 11111.... French, 1′ = 0,018518.... deg. French, and 1″ = 0,000308.... deg. French.
4) What do 57° ,9467 of the French division amount to in our degrees, minutes, and seconds? Ans. 52° 9′, 7″,308.
5) But what do 43°, 6′, 20″ of our division come to by the French system? Ans. 47°, 895....
6) What is the fourth proportional to the numbers 0·45, 0·8 and 0·367? Ans. 0,652444....
7) A cubic inch of the purest gold, weighs about 19·64 times as much as a cubic inch of distilled water; but a cubic inch of Japan copper only 9 times as much. How large must a piece of Japan copper be, to weigh as much as ¾ cubic inch of the gold? Ans. 1,6366.... cubic inches.

II. ALGEBRA.

The great progress of this science during the last centuries; the consequent increase and intricacy of its rules and expressions, and the limited power of our minds, soon made it necessary to introduce a language of symbols more comprehensive, expressive, and determinate, than mere words. How does Algebra, or the Literal Calculus, effect this object? What signs are used for Addition, Subtraction, Multiplication, and Division? What do you understand by an aggregate of quantities? What is the use of Brackets, or Crotchets? In what case are they used? Can they be omitted without mistake? What are coefficients?

What are Positive and Negative quantities? That which makes quantities opposite or contrary, is only a quality — a reference of one to the other. What is a magnitude without such reference? The Addition of opposite quantities is the Subtraction of absolute ones: Why?

* * *

1) ADDITION.

a) ADDITION OF SIMPLE QUANTITIES.

EXAMPLES.

1) $\begin{array}{r} a \\ a \\ \hline 2a \end{array}$
2) $\begin{array}{r} 7a \\ 5a \\ \hline 12a \end{array}$
3) $\begin{array}{r} 8f \\ f \\ \hline 9f \end{array}$
4) $\begin{array}{r} a \\ b \\ \hline a+b \end{array}$

5) $\begin{array}{r} 7a \\ 10x \\ \hline 7a+10x \end{array}$
6) $\begin{array}{r} a \\ -a \\ \hline 0 \end{array}$
7) $\begin{array}{r} 17a \\ -6a \\ \hline 11a \end{array}$
8) $\begin{array}{r} 5a \\ -9a \\ \hline -4a \end{array}$

9) $\begin{array}{r} -6a \\ 10a \\ \hline 4a \end{array}$
10) $\begin{array}{r} -3a \\ 2a \\ \hline -a \end{array}$
11) $\begin{array}{r} a \\ -b \\ \hline a-b \end{array}$
12) $\begin{array}{r} 8a \\ -5b \\ \hline 8a-5b \end{array}$

13) $\begin{array}{r} -7a \\ b \\ \hline -7a+b \\ \text{or } b-7a \end{array}$
14) $\begin{array}{r} -a \\ -a \\ \hline -2a \end{array}$
15) $\begin{array}{r} -3a \\ -8a \\ \hline -11a \end{array}$
16) $\begin{array}{r} -8a \\ -3b \\ \hline -8a-3b \\ \text{or} -(8a+3b) \end{array}$

17) $\begin{array}{r} 3a \\ -5a \\ 8a \\ \hline 6a \end{array}$
18) $\begin{array}{r} -12b \\ -\ 4b \\ 13b \\ \hline -3b \end{array}$
19) $\begin{array}{r} -6f \\ 9f \\ 13f \\ -8f \\ \hline 8f \end{array}$
20) $\begin{array}{r} -7c \\ -5c \\ -3c \\ \hline -15c \end{array}$

21) $\begin{array}{r} 5d \\ 7d \\ d \\ -8d \\ \hline 5d \end{array}$
22) $\begin{array}{r} 8e \\ -4e \\ 7e \\ -11e \\ \hline 0 \end{array}$
23) $\begin{array}{r} 3g \\ 2h \\ -5g \\ 7h \\ -10g \\ \hline -12g+9h \\ \text{or } 9h-12g \end{array}$
24) $\begin{array}{r} 3a \\ -5a \\ 7a \\ 2b \\ -c \\ 4b \\ \hline 5a+6b-c \end{array}$

b. ADDITION OF COMPOUND QUANTITIES.

1)
$$\begin{array}{rrr}
7a & -5c & +3b \\
2a & -3c & -7b \\
\hline
9a & -8c & -4b
\end{array}$$

2)
$$\begin{array}{rrrrr}
5a & +4b & -3c & -7d & +8 \\
3a & -12b & +7c & -10d & -4 \\
\hline
8a & -8b & +4c & -17d & +4
\end{array}$$

3)
$$\begin{array}{rrrrr}
12h & -3c & -7f & +3g & \\
-3h & +8c & -2f & -9g & +5x \\
\hline
9h & +5c & -9f & -6g & +5x
\end{array}$$

4)
$$\begin{array}{rrrrrr}
16a & -5b & +15c & -9d & & \\
3a & +18b & -5c & -7d & +3e & \\
-7a & -2b & & -3d & +5e & -9h \\
11a & -3b & +2c & +8d & & +7h \\
\hline
23a & +8b & +12c & -11d & +8e & -2h
\end{array}$$

5)
$$\begin{array}{rrrrr}
7x & -6y & +5z & +3 & -g \\
-x & -3y & & -8 & -g \\
-x & +y & -3z & -1 & +7g \\
-2x & +3y & +3z & -1 & -g \\
x & +8y & -5z & +9 & +g \\
\hline
4x & +3y & & +2 & +5g
\end{array}$$

6)
$$\begin{array}{rrrr}
8a & +b & & \\
2a & -b & +c & \\
-3a & +5b & & +2d \\
 & -6b & -3c & +3d \\
-5a & & +7c & -2d \\
\hline
2a & -b & +5c & +3d
\end{array}$$

7)
$$\begin{array}{rrrrrrr}
-a & +3b & -c & -115d & +6e & -5f & \\
3a & -2b & -3c & -d & +27e & & \\
 & 5b & -8c & & +3e & -7f & \\
-7a & -6b & +17c & +9d & -5e & +11f & \\
-3a & & -5c & -2d & +6e & -9f & +g \\
\hline
-8a & & & -109d & +37e & -10f & +g
\end{array}$$

8)
$$\begin{array}{rr}
-7f & +3a \\
4f & -2a \\
3f & -3a \\
 & +2a \\
\hline
 & 0
\end{array}$$

The teacher will do well to substitute in these examples, numbers instead of the letters, in order to illustrate to his pupils the correctness of the results obtained by Algebraic processes. This ought always to be done with beginners.

2) SUBTRACTION.

a. SUBTRACTION OF SIMPLE QUANTITIES.

EXAMPLES.

$$1)\ \begin{array}{r} a \\ a \\ \hline 0 \end{array} \qquad 2)\ \begin{array}{r} 7a \\ 3a \\ \hline 4a \end{array} \qquad 3)\ \begin{array}{r} 10f \\ 10f \\ \hline 0 \end{array} \qquad 4)\ \begin{array}{r} 5d \\ 11d \\ \hline -6d \end{array}$$

$$5)\ \begin{array}{r} 7a \\ 5b \\ \hline 7a-5b \end{array} \qquad 6)\ \begin{array}{r} a \\ -a \\ \hline 2a \end{array} \qquad 7)\ \begin{array}{r} 8a \\ -a \\ \hline 9a \end{array} \qquad 8)\ \begin{array}{r} 6a \\ -5a \\ \hline 11a \end{array}$$

$$9)\ \begin{array}{r} a \\ -4a \\ \hline 5a \end{array} \qquad 10)\ \begin{array}{r} a \\ -b \\ \hline a+b \end{array} \qquad 11)\ \begin{array}{r} 3a \\ -2b \\ \hline 3a+2b \end{array} \qquad 12)\ \begin{array}{r} -9a \\ 3a \\ \hline -12a \end{array}$$

$$17)\ \begin{array}{r} -7a \\ -7a \\ \hline 0 \end{array} \qquad 18)\ \begin{array}{r} -19a \\ -20a \\ \hline a \end{array} \qquad 19)\ \begin{array}{r} -6a \\ -5a \\ \hline -a \end{array} \qquad 20)\ \begin{array}{r} -3a \\ -5b \\ \hline -3a+5b \\ \text{or } 5b-3a \end{array}$$

$$21)\ \begin{array}{r} -13 \\ 3 \\ \hline -16 \end{array} \qquad 22)\ \begin{array}{r} -8 \\ -17 \\ \hline 9 \end{array} \qquad 23)\ \begin{array}{r} 12 \\ -7 \\ \hline 19 \end{array} \qquad 24)\ \begin{array}{r} -13 \\ -8 \\ \hline -5 \end{array}$$

b. SUBTRACTION OF COMPOUND QUANTITIES.

1) $$\begin{array}{r} 3a-2b+6 \\ 2a-7b-3 \\ \hline a+5b+9 \end{array}$$

2) $$\begin{array}{r} 13a-2b+9c-3d \\ 8a-6b+9c-10d+12 \\ \hline 5a+4b\quad+7d-12 \end{array}$$

3) $$\begin{array}{r} -7f+3m-8x \\ -6f-5m-2x+3d+8 \\ \hline -f+8m-6x-3d-8 \end{array}$$

4) $$\begin{array}{r} 2a-e-h-l \\ 9a-3e+4h-l-c \\ \hline -7a+2e-5h\quad+c \end{array}$$

5) $$\begin{array}{r} -a-5b+7c-d \\ 4b-3c+2d+3k \\ \hline -a-9b+10c-3d-3k \end{array}$$

6) $$\begin{array}{r} 3h-2k \\ 9l-7-8k \\ \hline 3h-9l+6k+7 \end{array}$$

7) $$\begin{array}{r} -3a+b-8c+7e-5f+3h-7x-13y \\ k+2a\quad-9c+8e+7f\quad-7x-y-3l-k \\ \hline -5a+b+c-e-12f+3h\quad-12y+3l \end{array}$$

8) $$\begin{array}{r} 5b-3a+203c+5 \\ -2b-8a+67c+7 \\ \hline 7b+5a+136c-2 \end{array}$$

9) $$\begin{array}{r} -14b+3c-27d+3-5g \\ 7a-5c-8d+3b-12+7g \\ \hline -7a-17b+8c-19d+15-12g \end{array}$$

10) $$\begin{array}{r} 6a+5-3b-5f-g-h \\ -2a-9b+8g-9h+7f-8 \\ \hline 8a+6b+13-12f-9g+8h \end{array}$$

11) $$\begin{array}{r} 3c-2l+5c \\ 8l+7c-4l \\ \hline c-6l \end{array}$$

12) $$\begin{array}{r} 3a-17b-10b+13a-2a \\ 6b-8a-b-2a+3d+9a-5h \\ \hline 15a-32b-3d+5h \end{array}$$

13) $$\begin{array}{r} 5c+3 \\ 2c-9-7c \\ \hline 10c+12 \end{array}$$

14) $$\begin{array}{r}8a-5b-3c-7d+5e-8f+3g+17k\\ \underline{-2k+3c-5b+2d-4e-7f+9g-5k-l}\\ 8a-6c\quad -9d+9e-f-6g+24k+l\end{array}$$

15) $32a+3b-(5a+17b)=27a-14b$

16) $13a-(5c+3f-7a-5x+3a)=17a-5c$
$-3f+5x$

17) $-8a+5b-3c-(7a-3b-2c)$
$=-15a+8b-c$

18) $3a-5c+3d-(7a-5d+8c-2e)$
$=8d+2e-4a-13c$

19) $37a-5f-(3a-2b-5c)-(6a-4b+3h)$
$=28a+6b-5f+5c-3h$

20) $a+b-(2a-3b)-(5a+7b)-(-13a+2b)$
$=7a-5b$

21) $\frac{1}{2}a-\frac{5}{6}x-(\frac{3}{4}a-\frac{1}{2}x)-(3b+\frac{11}{4}x-\frac{2}{3}a)$
$=\frac{5}{12}a-\frac{37}{12}x-3b=\frac{5a}{12}-\frac{37x}{12}-3b$

3) MULTIPLICATION.

a. MULTIPLICATION OF SIMPLE QUANTITIES.

EXAMPLES.

1) $a \times b = b \times a = ab = ba = a \,.\, b = b \,.\, a$

2) $a \times b \times c = abc = acb = bac = bca = cab = cba$

3) $-a \times b = -ab$

4) $a \times -b = -ab$

5) $-a \times -b = ab$

6) $6a \times 7b = 42ab$

7) $17a \times \frac{3}{4}b = \frac{51}{4}\,ab = \dfrac{51ab}{4}$

8) $\dfrac{3a}{2} \times \dfrac{5f}{4} = \dfrac{15af}{8} = \frac{15}{8}af$

9) $-3a \times 14c = -42ac$

10) $7a \times -10b = -70ab$

11) $\frac{3}{8}a \times -\frac{2}{3}b = -\frac{1}{4}ab = -\dfrac{ab}{4}$

12) $a \times -7b = -7ab$

13) $-6a \times -11x = 66ax$

14) $-\dfrac{5a}{4} \times -\dfrac{3b}{7} = \dfrac{15ab}{28}$

15) $ab \times cde = abcde$

16) $-5abc \times -7ade = 35aabcde$

17) $-5bd \times 9\,bbdxy = -45bbbddxy$

18) $-\frac{1}{2}abd \times \frac{1}{4}cdef = -\frac{1}{8}\,abcddef$

19) $17ace \times 5 = 85ace$

20) $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$

21) $-\frac{3fh}{5cde} \times ag = -\frac{3afgh}{5cde}$

22) $\frac{1}{fgh} \times 4cd = \frac{4cd}{fgh}$

23) $\frac{5ac}{bde} \times -\frac{7bfg}{3ad} = -\frac{35cfg}{3dde}$

24) $-\frac{h}{2fg} \times -\frac{g}{3h} = \frac{1}{6f}$

25) $3ab \times 2cd \times dfg = 6abcddfg$

26) $-\frac{a}{b} \times b \times -5df = 5adf$

27) $-4ab \times -\frac{3\,cde}{2aab} \times -\frac{1}{5cf} = -\frac{6de}{5af}$

28) $-a \times -a \times -a \times -a = aaaa$

29) $\frac{2ac}{bg} \times -\frac{3bd}{4cfh} \times hf = -\frac{3ad}{2g}$

30) $\frac{1}{a} \times \frac{3ac}{x} \times \frac{c}{x} = \frac{3cc}{xx}$

31) $\frac{2fg}{bdh} \times \frac{3xy}{7fz} \times -\frac{bc}{5d} = -\frac{6cgxy}{35ddhz}$

b. MULTIPLICATION OF COMPOUND QUANTITIES.

1) $(6a + 3b - 5f) \times 5g = 30ag + 15bg - 25fg$

2) $(-2b + 3c - g) \times -8h = 16bh - 24ch + 8gh$

3) $(7ad - 15bc - 16acf) \times 10ab = 70aabd - 150abbc$
$- 160aabcf$

4) $\left(\frac{5a}{b} - \frac{13c}{2d} - \frac{6h}{5bg} + 7d\right) \times \frac{3a}{5d} = \frac{3aa}{bd} - \frac{39ac}{10dd}$
$- \frac{18ah}{25bdg} + \frac{21a}{5}$

5) $(a + b)(c + d) = ac + bc + ad + bd$

6) $(a + b - c)(d - e) = ad + bd - cd - ae - be + ce$

7) $(2a - 3b - 8c - d + 9e) \times (7f + 2g - h) =$
$14af - 21bf - 56cf - 7df + 63ef + 4ag - 6bg$
$- 16cg - 2dg + 18eg - 2ah + 3bh + 8ch + dh$
$- 9eh$

8) $(7l - 2m - 9) \times (3l - 11m) = 21ll - 83lm - 27l$
$+ 22mm + 99m$

9) $(2a + 5b + 3c - 5e) \times (3a + 10b + 15f) = 6aa$
$+ 35ab + 9ac - 15ae + 50bb + 30bc - 50be$
$+ 30af + 75bf + 45cf - 75ef$

10) $(a + b) \times (a - b) = aa - bb$

11) $(a + b) \times (a + b) = aa + 2ab + bb$

12) $(a - b) \times (a - b) = aa - 2ab + bb$

13) $(3a + 5b - \frac{7}{2}c) \times (a - 2b + 9c) = 3aa - ab$
$+ \frac{47}{2}ac - 10bb + 52bc - \frac{63}{2}cc$

14) $(3c - 5d + \frac{3}{4}g - \frac{5}{3}h) \times (\frac{2}{3}c - d + 7g + \frac{1}{2}h) =$
$2cc - \frac{19}{3}cd + \frac{43}{2}cg + \frac{7}{18}ch + 5dd - \frac{143}{4}dg - \frac{5}{6}dh$
$+ \frac{21}{4}gg - \frac{271}{24}gh - \frac{5}{6}hh$

15) $(5ab + 3ac - 4bc) \times (7ab - 18ac + 2bc + d)$
$= 35aabb - 69aabc - 18abbc + 5abd - 54aacc$
$+ 78abcc + 3acd - 8bbcc - 4bcd$

16) $(13bcd + 20bce - 10bde) \times (4bc + 3bd - 12e)$
$= 52bbccd + 80bbcce + 20bbcde + 39bbcdd$
$- 30bbdde - 156bcde - 240bcee + 120bdee$

17) $(5aa - 3ab + 7bb) \times (3a - b) = 15aaa - 14aab$
$+ 24abb - 7bbb$

18) $(a + b + c) \times (a + b - c) = aa + 2ab + bb - cc$

19) $(3aaa + 35aab - 17abb - 13bbb) \times (3aa + 26ab$
$- 57bb) = 9aaaaa + 183aaaab + 688aaabb$
$- 2476aabbb + 631abbbb + 741bbbbb$

20) $(3a - b + 2c - 3d + 5e) \times (17a - 2b + 12c)$
$= 51aa - 23ab + 70ac - 51ad + 85ae + 2bb$
$- 16bc + 6bd - 10be + 24cc - 36cd + 60ce$

21) $(a + b + c + d) \times (a - b - c - d) = aa - bb$
$- 2bc - 2bd - cc - 2cd - dd$

22) $(- 2a + 3b - cc) \times (- 3f - 7a + cc) = 6af$
$- 9bf + 3ccf + 14aa - 21ab + 5acc + 3bcc$
$- cccc$

23) $(\frac{3}{2}m - 5n - \frac{1}{3}pp) \times (\frac{1}{4}m - 2n + 6pp) = \frac{3}{8}mm$
$- \frac{17}{4}mn + \frac{107}{12}mpp + 10nn - \frac{88}{3}npp - 2pppp$

24) $\left(\frac{5ff}{g} - \frac{7fg}{4h} + 3f\right) \times \left(\frac{7g}{f} + \frac{2f}{g}\right) = 35f - \frac{49gg}{4h}$
$+ 21g + \frac{10fff}{gg} - \frac{7ff}{2h} + \frac{6ff}{g}$

25) $\left(\frac{aa}{xx} - \frac{ab}{2xy} + \frac{bb}{yy}\right) \times \left(\frac{3aa}{xx} - \frac{2ab}{5xy} + \frac{bb}{yy}\right) =$
$\frac{3aaaa}{xxxx} - \frac{19aaab}{10xxxy} + \frac{21aabb}{5xxyy} - \frac{9abbb}{10xyyy} + \frac{bbbb}{yyyy}$

4) DIVISION.

a. DIVISION OF SIMPLE QUANTITIES.

EXAMPLES.

1) $a : b = \frac{a}{b}$

2) $-a : b = -\frac{a}{b}$

3) $a : -b = -\frac{a}{b}$

4) $-a : -b = \frac{a}{b}$

5) $5a : 3b = \frac{5a}{3b}$

6) $-16a : 8b = -\frac{2a}{b}$

7) $12a : -4b = -\frac{3a}{b}$

8) $-14a : -4b = \frac{7a}{2b}$

9) $abc : a = bc$

10) $abc : ad = \frac{bc}{d}$

11) $8fmn : -2fgm = -\frac{4n}{g}$

12) $-12abcde : -8acd = \frac{3be}{2}$

13) $6abde : -2bf = -\frac{3ade}{f}$

14) $27aaabbcfg : -18abcghk = -\frac{3aabf}{2hk}$

15) $35abfgm : 5aabffgmn = \frac{7}{afn}$

16) $\frac{a}{b} : \frac{c}{d} = \frac{ad}{bc}$

17) $\frac{3afx}{bc} : \frac{2fxx}{5cde} = \frac{15ade}{2bx}$

18) $3fm : -\frac{3am}{5bg} = -\frac{5bfg}{a}$

19) $\frac{2ay}{5bcx} : 3ac = \frac{2y}{15bccx}$

20) $\frac{1}{7fggl} : \frac{1}{4fln} = \frac{4n}{7gg}$

21) $\frac{3}{4}ac : \frac{5}{6}abd = \frac{9c}{10bd}$

b. DIVISION OF COMPOUND QUANTITIES.

1) $\left(3ac - 2ade - f + \frac{c}{d}\right) : 2a = \frac{3c}{2} - de - \frac{f}{2a} + \frac{c}{2ad}$

2) $(18acf - 6bdef - 2ad) : 3adf = \frac{6c}{d} - \frac{2be}{a} - \frac{2}{3f}$

3) $(8aa - 6ab + 4c + 1) : -2a = -4a + 3b - \frac{2c}{a} - \frac{1}{2a}$

4) $(12acfg - 4affg + 3fggh) : 4aabbfg = \frac{3c}{abb} - \frac{f}{abb} + \frac{3gh}{4aabb}$

5) $\left(\frac{a}{b} + \frac{fd}{2c} - 3ac + 7\right) : \frac{3c}{d} = \frac{ad}{3bc} + \frac{fdd}{6cc} - ad + \frac{7d}{3c}$

6) $(ab - ac) : (b - c) = a$

7) $(ac - bc + ad - bd) : (a - b) = c + d$

8) $(4aa + 6ab - 4ax + 9bx - 15xx) : (2a + 3x) = 2a + 3b - 5x$

9) $(14af - 21bf + 7cf + 6ag - 9bg + 3cg) : (7f + 3g) = 2a - 3b + c$

10) $(4xxx + 4xx - 29x + 21) : (2x - 3) = 2xx + 5x - 7$

11) $(\frac{3}{2}xxx - \frac{5}{4}xx - 8x + 9) : (\frac{1}{2}x - 1) = 3xx + \frac{7}{2}x - 9$

12) $(aa + ab + 2ac - 2bb + 7bc - 3cc) : (a + 2b - c) = a - b + 3c$

13) $(12aa + 26ab - 36ac + 18ad - 10bb + 29bc - 6bd - 21cc + 9cd) : (6a - 2b + 3c) = 2a + 5b - 7c + 3d$

14) $(119cc - 200cd + 408ce - 113cf - 39dd + 72de + 37df - 96ef + 20ff) : (17c + 3d - 4f) = 7c - 13d + 24e - 5f$

15) $\left(3aa - \frac{7ab}{2} - \frac{21ac}{4} - \frac{5bb}{2} + \frac{83bc}{8} - \frac{3cc}{2}\right) : \left(3a - 5b + \frac{3c}{4}\right) = a + \frac{b}{2} - 2c$

16) $\left(2ff - \frac{55fh}{12} + \frac{29fx}{9} + \frac{21hh}{8} - \frac{15hx}{4} + \frac{xx}{3}\right) : \left(\frac{2f}{3} - \frac{3h}{4} + x\right) = 3f - \frac{7h}{2} + \frac{x}{3}$

17) $(30aab - 6aac + 75abb - 15abc) : (15ab - 3ac) = 2a + 5b$

18) $(36aab - 63abb + 20bbb) : (12ab - 5bb) = 3a - 4b$

19) $(72xxxx - 78xxxy - 10xxyy + 17xyyy + 3yyyy) : (6xx - 4xy - yy) = 12xx - 5xy - 3yy$

20) $(\frac{1}{3}xxxx - \frac{11}{12}xxx + \frac{41}{8}xx - \frac{23}{4}x + 6) : (\frac{2}{3}xx - \frac{5}{6}x + 1)$
$= \frac{1}{2}xx - \frac{3}{4}x + 6$

21) $\left(\frac{15aa}{2} - \frac{113ab}{6} + 9ac + 2bb - bc\right) :$
$\left(\frac{5a}{4} - 3b + \frac{3c}{2}\right) = 6a - \frac{2b}{3}$

22) $\left(-\frac{5xx}{9} + \frac{11xy}{3} - \frac{10xz}{3} + \frac{15yy}{4} + 25yz\right) :$
$(-\frac{2}{3}x + 5y) = \frac{5x}{6} + \frac{3y}{4} + 5z$

23) $(18aa + 33ab + 42ac - 12ad - 30bb + 124bc$
$+ 8bd - 16cc - 32cd) : (6a + 15b - 2c - 4d)$
$= 3a - 2b + 8c$

24) $(108aa - 33ac - 9ad - 9ae - 24ab + 10bc + 2bd$
$+ 2be - 5cc - cd - ce) : (12a - 5c - d - e)$
$= 9a - 2b + c$

25) $\left(\frac{5}{2}xy + \frac{85}{9}yy - \frac{223}{54}yz - \frac{10}{9}yu - \frac{3}{4}xz + \frac{7}{18}zz + \frac{uz}{3}\right)$
$: (\frac{3}{2}x + \frac{17}{3}y - \frac{7}{9}z - \frac{2}{3}u) = \frac{5}{3}y - \frac{1}{2}z$

26) $(- 75aabfxx + 65aacxy + 60abxyy - 65axz$
$+ 180abfxxy - 156acxyy - 144bxyyy + 156xyz)$
$: (15abfx - 12byy - 13acy + 13z) = - 5ax$
$+ 12xy$

27) $\left(\frac{aa}{bc} - \frac{2a}{d} + \frac{ac}{be} + \frac{bc}{dd} - \frac{cc}{de}\right) : \left(\frac{a}{b} - \frac{c}{d}\right) =$
$\frac{a}{c} - \frac{b}{d} + \frac{c}{e}$

28) $\left(\frac{5aax}{b} - \frac{5aby}{c} + 5ad - ax + \frac{bby}{c} - bd\right) :$
$\left(\frac{ax}{b} - \frac{by}{c} + d\right) = 5a - b$

29) $\left(\frac{aaxxx}{bd} + \frac{abxx}{ccd} - \frac{acxx}{dd} - \frac{bbx}{cdd} + \frac{aax}{bc} - \frac{a}{d}\right)$
$: \left(\frac{ax}{c} - \frac{b}{d}\right) = \frac{acxx}{bd} + \frac{bx}{cd} + \frac{a}{b}$

30) $(aa - bb) : (a - b) = a + b$

31) $(aaaa - 9aabb - 6abcc - cccc) : (aa - 3ab - cc)$
$= aa + 3ab + cc$

32) $(aaaa - bbbb) : (a - b) = aaa + aab + abb + bbb$

33) $(32aaaaa + bbbbb) : (2a + b) = 16aaaa - 8aaab$
$+ 4aabb - 2abbb + bbbb$

34) $\left(\frac{9aabb}{4cc} - \frac{25ffmm}{gg} + \frac{70dfm}{g} - 49dd\right) :$
$\left(\frac{3ab}{2c} + \frac{5fm}{g} - 7d\right) = \frac{3ab}{2c} - \frac{5fm}{g} + 7d$

c. PARTIAL DIVISION, PERFORMED IN CASES WHERE THE DIVISOR IS NOT AN EXACT NUMBER OF TIMES CONTAINED IN THE DIVIDEND.

1) $$1 : (1 - b) = 1 + \frac{b}{1 - b}$$

$$= 1 + b + \frac{bb}{1 - b}$$

$$= 1 + b + bb + \frac{bbb}{1 - b}$$

$$= 1 + b + bb + bbb + \frac{bbbb}{1 - b}$$

$$= 1 + b + bb + bbb + bbbb + \ldots.$$

2) $$1 : (1 + b) = 1 - \frac{b}{1 + b}$$

$$= 1 - b + \frac{bb}{1 + b}$$

$$= 1 - b + bb - \frac{bbb}{1 + b}$$

$$= 1 - b + bb - bbb + \frac{bbbb}{1 + b}$$

$$= 1 - b + bb - bbb + bbbb - \ldots\ldots$$

3) $$c : (a - b) = \frac{c}{a} + \frac{bc}{a(a-b)}$$
$$= \frac{c}{a} + \frac{bc}{aa} + \frac{bbc}{aa(a-b)}$$
$$= \frac{c}{a} + \frac{bc}{aa} + \frac{bbc}{aaa} + \frac{bbbc}{aaa(a-b)}$$
$$= \frac{c}{a} + \frac{bc}{aa} + \frac{bbc}{aaa} + \frac{bbbc}{aaaa} + \dots$$

4) $$c : (a + b) = \frac{c}{a} - \frac{bc}{a(a+b)}$$
$$= \frac{c}{a} - \frac{bc}{aa} + \frac{bbc}{aa(a+b)}$$
$$= \frac{c}{a} - \frac{bc}{aa} + \frac{bbc}{aaa} - \frac{bbbc}{aaa(a+b)}$$
$$= \frac{c}{a} - \frac{bc}{aa} + \frac{bbc}{aaa} - \frac{bbbc}{aaaa} + \dots$$

5) $$(1 + x) : (1 - x) = 1 + \frac{2x}{1-x}$$
$$= 1 + 2x + \frac{2xx}{1-x}$$
$$= 1 + 2x + 2xx + \frac{2xxx}{1-x}$$
$$= 1 + 2x + 2xx + 2xxx + \dots$$

III. ADDITION, SUBTRACTION, MULTIPLICATION AND DIVISION OF POWERS.

Questions to be answered. What is meant by the Powers of a quantity? What, by the Exponent of a Power? What is the Basis of a Power? Is the value of a Power altered when its Basis and Exponent are interchanged? How are Powers of the same Basis multiplied and divided? How, when they have different Bases? Should we, in Division, meet with powers whose exponents are 0, or negative, what meaning have such Powers? Are the former rules of Multiplication and Division applicable to such powers? The Exponent 1, may be either omitted or expressed. How is Addition and Subtraction of Powers performed? How can you transpose a quantity from the numerator of a fraction to the denominator, and vice versa? What change takes place in the Exponent of a Power, when that Power is removed from the Numerator to the Denominator of a Fraction, and vice versa.

1) ADDITION AND SUBTRACTION.

EXAMPLES.

1) $ax^n + bx^n + cx^n + dx^n = (a + b + c + d)\,x^n$

2) $ax^n + bx^n - cx^n - dx^n = (a + b - c - d)\,x^n$

3) $10a^4 + 3a^4 + 6a^4 - a^4 - 5a^4 = 13a^4$

4) $3a^{-7} + 10a^{-7} - 5a^{-7} + a^2b = 8a^{-7} + a^2b$

5) $6^4 + 2\,.\,8^3 + 3^2 - 19\,.\,6^4 + 5\,.\,8^3 = 7\,.\,8^3$
$- 18\,.\,6^4 + 3^2$

6) $16a^4b^3c^5 - 6a^4b^3c^5 + 7a^4b^3c^5 + 17a^4b^3c^5$

7) $\dfrac{5a^3}{b^4} - \dfrac{7a^3}{b^4} + \dfrac{11a^3}{b^4} + a^4 = \dfrac{9a^3}{b^4} + a^4$

8) $a^nb^m - 9a^m + 5a^nb^m + 6a^m + 10a^nb^m = 16a^nb^m - 3a^m$

9) $5a^{-3}b^2 + 7ab^2c - 3a^mb^{-5} - 12ab^2c + 6a^{-3}b^2 - 9a^3b^3$
$+ b^{-x} - 8a^mb^{-5} - 3b^{-x} = 11a^{-3}b^2 - 5ab^2c$
$- 11a^mb^{-5} - 9a^3b^3 - 2b^{-x}$

10) $3\,.\,2^{-7} + 5^6 - 8\,.\,2^{-7} + 3a^nb^{-m} - 13\,.\,5^6 + 4a\,.\,2^{-7}$
$+ ca^nb^{-m} = (4a - 5)\,2^{-7} - 12\,.\,5^6 + (c + 3)\,a^nb^{-m}$

11) For Addition.
$$\left\{\begin{array}{r}
5a^4b + 3a^{-2}b^2c - 7ab \\
-\,6a^4b + 2a^{-2}b^2c + 17ab \\
9a^4b - 8a^{-2}b^2c - 10ab \\
\hline
8a^4b - 3a^{-2}b^2c
\end{array}\right.$$

$$12)\quad \text{For Add.}\left\{\begin{array}{l} 5a^mb^p + 3a^{-3}b^{m-1} - \dfrac{3a^3}{x^p} \\ -3ca^mb^p + 4g^2a^{-3}b^{m-1} - a + \dfrac{10a^3}{x^p} \\ a^mb^p + a + 3a^2b^2 - 2g^2a^{-3}b^{m-1} \\ \hline (6-3c)a^mb^p + (2g^2+3)a^{-3}b^{m-1} + \dfrac{7a^3}{x^p} + 3a^2b^2 \end{array}\right.$$

$$13)\quad \text{For Add.}\left\{\begin{array}{l} 9a^{-3}b^{-2}c^4 - \dfrac{7b}{a^3} \\ \dfrac{18b}{a^3} - 5a^n b^m + c^x - 3 . 2^5 \\ 3a^nb^m - ha^{-3}b^{-2}c^4 + 3c^x - 5 . 2^5 \\ \hline (9-h)\, a^{-3}b^{-2}c^4 - 2a^nb^m + \dfrac{11b}{a^3} + 4c^x - 8.2^5 \end{array}\right.$$

$$14)\quad \text{For Subt.}\left\{\begin{array}{l} 9a^mx^2 - 13 + 20ab^3x - 4b^mcx^2 \\ 3b^mcx^2 + 9a^mx^2 - 6 + 3ab^3x \\ \hline 17ab^3x - 7b^mcx^2 - 7 \end{array}\right.$$

$$15)\quad \text{For Subt.}\left\{\begin{array}{l} 5a^4 - 7a^3b^2 - 3c^{-1}d^2 + 7d \\ 3a^4 - 15a^3b^2 - 7c^{-1}d^2 - 3a^2 \\ \hline 2a^4 + 8a^3b^2 + 4c^{-1}d^2 + 7d + 3a^2 \end{array}\right.$$

2) MULTIPLICATION.

a. MULTIPLICATION OF SIMPLE QUANTITIES.

EXAMPLES.

1) $a^m \times a^n = a^{m+n}$

2) $a^{-m} \times a^n = a^{-m+n} = a^{n-m}$

3) $a^m \times a^{-n} = a^{m-n}$

4) $a^{-m} \times a^{-n} = a^{-m-n} = a^{-(m+n)}$

5) $5a^3 \times a^7 \times 7a^5 \times 3a^6 = 105a^{21}$

6) $11a^{-2} \times 2a^{-5} \times 4a^6 \times 9a^7 = 792a^6$

7) $2a^{-3} \times 7a^{-9} \times - 3a^6 = - 42a^{-6} = - \frac{42}{a^6}$

8) $a^{-5}b \times a^{-7}d \times 10a = 10a^{-11}bd = \frac{10bd}{a^{11}}$

9) $3 \,.\, 7^{-9} \times 7^{-2} \times 4 \,.\, 7^8 = 12 \,.\, 7^{-3} = \frac{12}{343}$

10) $- a^{p-q} \times - 3a^{3q-2}f \times 5a^{p+7}cx = 15a^{2p+2q+5}fcx$

11) $5a^3b^{-4} \times 10a^2b^5c \times - 3a^7 = - 150a^{12}bc$

12) $- 7a^{-1}b^4c^{-5} \times 3a^2b^{-5}c = - 21ab^{-1}c^{-4} = - \frac{21a}{bc^4}$

13) $5a^3b^4 \times a^2b^8 \times 4acb^{-3} = 20a^6b^9c$

14) $h^4l^{11}x \times h^{-7}l^{-9} \times 3h^{-2}l^{-6}x^3 = 3h^{-5}l^{-4}x^4 = \frac{3x^4}{h^5l^4}$

15) $- 13a^{-1}c^{-3} \times - 4a^{-3}b^{-6}c^2 = 52a^{-4}b^{-6}c^{-1} = \frac{52}{a^4b^6c}$

16) $- \frac{5}{2}a^{-2}b^3c^{-m}d^{-1} \times \frac{2}{3}a^2b^{-3}c^{-2}d^4 = - \frac{5}{3}c^{-m-2}d^3$
$= - \frac{5d^3}{3c^{m+2}}$

17) $a^{-m}b^pc^q \times a^nb^{-r}c^s \times a^{n+m}b = a^{2n}\, b^{p-r+1}c^{q+s}$

18) $f^xg^{m-1}hd \times 2f^{m+3}h^{2-m} \times 4g^{2m+1}h^{m+2}d^3 =$
$8f^{x+m+3}g^{3m}h^5d^4$

19) $(a + y)^{-3}h^5l^4 \times (a + y)^{m+3}l^{-4}m \times (a + y) =$
$mh^5\,(a + y)^{m+1}$

20) $\frac{18a^{-5}b^3}{7c^{-2}d^{-6}} \times \frac{4a^6b^{-5}}{9c^3d^9} = \frac{72ab^{-2}}{63ca^3} = \frac{8a}{7cd^3b^2}$

21) $\frac{6a^4b^5c^{-7}}{11f^3dg^{-4}} \times \frac{3a^{-2}b^4c^{-1}}{5g^2f^6} = \frac{18a^2b^9c^{-8}}{55f^9dg^{-2}} = \frac{18a^2b^9g^2}{55f^9dc^8}$

22) $\frac{1}{3a^{-3}b^{-m}c} \times \frac{1}{4a^{-p}b} = \frac{1}{12a^{-p-3}b^{-m+1}c} = \frac{a^{p+3}b^{m-1}}{12c}$

23) $\frac{2a^{-m-3}b^{m+2}}{3x^{-5}y^{-n}z^p} \times \frac{6a^{-1}b^{-m}}{x^{-p}y^3} = \frac{12a^{-m-4}b^2}{3x^{-p-5}y^{-n+3}z^p} =$
$\frac{4b^2x^{p+5}y^{n-3}}{a^{m+4}z^p}$

24) $\frac{3a^{-5}b^2c^3f^4}{(a+b)^{-m}(c^2+x^2)} \times \frac{a^2b^{-5}c^{-2}f^{-4}}{(a+b)^{m-2}(c^2+x^2)^{-n+4}}$
$= \frac{3a^{-3}b^{-3}c}{(a+b)^{-2}(c^2+x^2)^{-n+5}} = \frac{3c(a+b)^2(c^2+x^2)^{n-5}}{a^3b^3}$

b. MULTIPLICATION OF COMPOUND QUANTITIES.

1) $(a^2 - 3ab - 5b^2) \times 4a^2b = 4a^4b - 12a^3b^2 - 20a^2b^3$

2) $(2a^3b^5 - 5a^2c^6 + 9a^3b^2c^3) \times 3a^2bc^2 = 6a^5b^6c^2$
$- 15a^4bc^8 + 27a^5b^3c^5$

3) $(7h^{-5}l + \frac{2l^3}{h^4} - 3ah^{-3}l^2 + 7) \times - 8h^4l^{-5} =$
$- 56h^{-1}l^{-4} - 16l^{-2} + 24ahl^{-3} - 56h^4l^{-5} =$
$\frac{24ah}{l^3} - \frac{56}{hl^4} - \frac{56h^4}{l^5} - \frac{16}{l^2}$

4) $\left(a^3b^{-4} - cb^{-5}d^3f + \frac{3c^m}{k^3}\right) \times 2bc^{-2}d = 2a^3b^{-3}c^{-2}d$
$-2b^{-4}c^{-1}d^4f + \frac{6bc^{m-2}d}{k^3} = \frac{2a^3d}{b^3c^2} - \frac{2d^4f}{b^4c} + \frac{6bc^{m-2}d}{k^3}$

5) $(a^{m-1}b^{2m+3} - 6a^{3-m}b^{p} + ab^{-m}) \times a^{3m+2}b^{m-1} = a^{4m+1}b^{3m+2}$
$- 6a^{2m+5}b^{p+m-1} + \frac{a^{3m+3}}{b}$

6) $(a + b) \times (a + b) = (a + b)^2 = a^2 + 2ab + b^2$

7) $(a - b) \times (a - b) = (a - b)^2 = a^2 - 2ab + b^2$

8) $(a + b) \times (a - b) = a^2 - b^2$

9) $(a^4 - 2b^3) \times (a - b) = a^5 - 2ab^3 - a^4b + 2b^4$

10) $(x^2 - 3x - 7) \times (x - 2) = x^3 - 5x^2 - x + 14$

11) $(3k^2 - 5kl + 2l^2) \times (k^2 - 7kl) = 3k^4 - 26k^3l$
$+ 37k^2l^2 - 14kl^3$

12) $(6f^2 - 17fl + 3l^2) \times (f^5 + 4f^4l) = 6f^7 + 7f^6l$
$- 65f^5l^2 + 12f^4l^3$

13) $(4a^2 - 16ax + 3x^2) \times (5a^3 - 2a^2x) = 20a^5$
$-88a^4x + 47a^3x^2 - 6a^2x^3$

14) $(a^2 + a^4 + a^6) \times (a^2 - 1) = a^8 - a^2$

15) $(a^4 - 2a^3b + 4a^2b^2 - 8ab^3 + 16b^4) \times (a + 2b)$
$= a^5 + 32b^5$

16) $(2a^4x^2 - 3b^4y^2) \times (2a^4x^2 + 3b^4y^2) = 4a^8x^4 - 9b^8y^4$

17) $(7a^3 - 5a^2b + 6ab^2 - 2b^3) \times (3a^4 - 4a^3b + 16a^2b^2)$
$= 21a^7 - 43a^6b + 150a^5b^2 - 110a^4b^3 = 104a^3b^4$
$- 32a^2b^5$

18) $(\frac{5}{2}x^2 + 3ax - \frac{7}{3}a^2) \times (2x^2 - ax - \frac{1}{2}a^2) = 5x^4$
$+ \frac{7}{2}ax^3 - \frac{107}{12}a^2x^2 + \frac{5}{6}a^3x + \frac{7}{6}a^4$

19) $(a^6 - 3a^4b^2 + 5a^2b^4) \times (7a^4 - 4a^2b^2 + b^4) = 7a^{10}$
$- 25a^8b^2 + 48a^6b^4 - 23a^4b^6 + 5a^2b^8$

20) $(a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5)$
$\times (a^3 - 3a^2b + 3ab^2 - b^3) = a^8 - 8a^7b + 28a^6b^2$
$- 56a^5b^3 + 70a^4b^4 - 56a^3b^5 + 28a^2b^6 - 8ab^7 + b^8$

21) $(a^2 + az + z^2) \times (a^2 - az + z^2) = a^4 + a^2z^2 + z^4$

22) $(15a^{-6}b^2 - 7a^{-5}b^4 + 6a^{-4}b^6) \times (8a^{-2}b^2 - 3a^{-1}b^4)$
$= 120a^{-8}b^4 - 101a^{-7}b^6 + 69a^{-6}b^8 - 18a^{-5}b^{10}$

23) $(13a^{-5}b + 10a^{-2}b^2 - 4ab^3) \times (6a^{-3}b^2 - 18b^3$
$- 7a^3b^4) = 78a^{-8}b^3 - 174a^{-5}b^4 - 295a^{-2}b^5$
$+ 2ab^6 + 28a^4b^7$

24) $(3x^{-2}y^{-7} - 2x^2y^{-5} + 8x^6y^{-3}) \times (2x^{-3}y^{-5} + 6xy^{-3}$
$+ 12x^5y^{-1}) = 6x^{-5}y^{-12} + 14x^{-1}y^{-10} + 40x^3y^{-8}$
$+ 24x^7y^{-6} + 96x^{11}y^{-4}$

25) $(5a^3b^3c^2 - 6a^4b^2c^5 + 7a^8b^5c^6) \times (2a^3b^3c^2 + 3a^4b^2c^5$
$- 6a^7b^4c^3) = 10a^6b^6c^4 + 3a^7b^5c^7 + 14a^{11}b^8c^8$
$- 18a^8b^4c^{10} + 21a^{12}b^7c^{11} - 30a^{10}b^7c^5 + 36a^{11}b^6c^8$
$- 42a^{15}b^9c^9$

26) $(14a^5c^2 - 6a^2bc^2 + c^3) \times (14a^5c^2 + 6a^2bc^2 - c^3)$
$= 196a^{10}c^4 - 36a^4b^2c^4 + 12a^2bc^5 - c^6$

27) $\left(\frac{a^2}{b^3} + \frac{2c^3d^4}{b^5} - \frac{7c^2}{2a^4b^3}\right) \times \left(\frac{a^2}{b^3} - \frac{2c^3d^4}{b^5} + \frac{7c^2}{2a^4b^3}\right)$
$= \frac{a^4}{b^6} - \frac{4c^6d^8}{b^{10}} + \frac{14c^5d^4}{a^4b^8} - \frac{49c^4}{4a^8b^6}$

28) $(a^m + b^p - 2c^n) \times (2a^m - 3b) = 2a^{2m} + 2a^mb^p$
$- 4a^mc^n - 3a^mb - 3b^{p+1} + 6bc^n$

29) $(2a^{3-2m}b^{n+3} + 3a^{m+1}b^{n+2} + c^p) \times (a^{m-1}b^{1-2m} - ca^p)$
$= 2a^{2-m}b^{n-2m+4} + 3a^{2m}b^{n-2m+3} + a^{m-1}b^{1-2m}c^p$
$- 2ca^{p-2m+3}b^{n+3} - 3ca^{p+m+1}b^{n+2} - a^pc^{p+1}$

30) $(x^{-3p} + 3a^mx^{-2p} - 10a^{2m}x^{-p}) \times (a^2x^q + 5a^{m+2}x^{q+p}$
$- 2a^{2m+2}x^{q+2p}) = a^2x^{q-3p} + 8a^{m+2}x^{q-2p}$
$+ 3a^{2m+2}x^{q-p} - 56a^{3m+2}x^q + 20a^{4m+2}x^{q+p}$

31) $(3a^{4-3m}bc^{m-2} + 17a^{-3}b^{m+1}) \times (3a^{6m-2}b^{2m}c^{3-4m} - 8)$
$= 9a^{3m+2}b^{2m+1}c^{1-3m} + 51a^{6m-5}b^{3m+1}c^{3-4m}$
$- 24a^{4-3m}bc^{m-2} - 136a^{-3}b^{m+1}$

The formulæ 6, 7, 8, (*b*) are highly important; how can they be expressed in words?

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3) DIVISION.

a. DIVISION OF SIMPLE QUANTITIES.

EXAMPLES.

1) $a^m : a^n = a^{m-n}$

2) $a^m : a^{-n} = a^{m+n}$

3) $a^{-m} : a^n = a^{-m-n} = a^{-(m+n)}$

4) $a^{-m} : a^{-n} = a^{n-m}$

5) $8a^{10} : 2a^4 = 4a^6$

6) $\frac{7}{3}a^3 : \frac{2}{5}a^7 = \frac{35}{6}a^{-4} = \frac{35}{6a^4}$

7) $\frac{14}{5}a^{-8} : -3a = -\frac{14}{15}a^{-9} = -\frac{14}{15a^9}$

8) $ca^{18} : da^{-6} = \frac{ca^{24}}{d}$

9) $6(a+b)^{-9} : 4(a+b)^{-5} = \frac{3}{2}(a+b)^{-4} = \frac{3}{2(a+b)^4}$

10) $\frac{5}{3}a^{-7}b^3c : \frac{1}{2}a^{-9}b^{-5}c^3f = \frac{10a^2b^8f}{3c^2}$

11) $(a+x)^2(a+y)^{-3} : (a+x)^{-4}(a+y)^{-7} =$
$(a+x)^6(a+y)^4$

12) $-3a^mb^n : -4a^pb^qc^r = \frac{3a^{m-p}b^{n-q}}{4c^r}$

13) $-\frac{5c^2a^{-m}b^n}{8} : 3cd^5a^pb^{-q} = -\frac{5cb^{n+q}}{24d^5a^{m+p}}$

14) $\frac{3a^3d}{2b^5} : \frac{b^3}{4a^2c^7} = \frac{6a^5c^7d}{b^8}.$

15) $\frac{2c^3(1+z^2)^2}{d^7z^9} : \frac{5c^8f^3(1+z^2)^{-6}}{2d^9z^5} = \frac{4d^2(1+z^2)^8}{5c^5f^3z^4}$

16) $\frac{2x^{3n-5m}y^{2n-3}}{7a^mb^3c} : \frac{4x^{1-5m}}{3ab^{n-1}y^5} = \frac{3b^{n-4}y^{2n+2}x^{3n-1}}{14a^{m-1}c}$

b. DIVISION OF COMPOUND QUANTITIES.

1) $(6a^3b^2 - 10a^2f + 7a^4bx) : 2a^2 = 3ab^2 - 5f + \frac{7}{2}a^2bx$

2) $(\frac{3}{4}a^2x^5 - \frac{7}{9}ax^3 + 3ab^2x) : \frac{2}{3}a^2x^3 = \frac{9}{8}x^2 - \frac{7}{6a} + \frac{9b^2}{2ax^2}$

3) $\left(\frac{3a^2b^6}{4} - \frac{5ac}{b^7} + 2a^3c^2 - \frac{2a^2c^2}{5b(a+y)^2}\right) : -\frac{2a^5b^2}{3c}$
$= -\frac{9b^4c}{8a^3} + \frac{15c^2}{2a^4b^9} - \frac{3c^3}{a^2b^2} + \frac{3c^3}{5a^3b^3(a+y)^2}$

4) $(bc^3 - c^3x) : (b - x) = c^3$

5) $(a^2 + 2ab + b^2) : (a + b) = a + b$

6) $(a^3 + a^2b - ab^2 - b^3) : (a - b) = a^2 + 2ab + b^2$

7) $(3a^5 + 16a^4b - 33a^3b^2 + 14a^2b^3) : (a^2 + 7ab)$
$= 3a^3 - 5a^2b + 2ab^2$

8) $(a^7 - 6a^6b^3 + 14a^5b^6 - 12a^4b^9) : (a^3 - 2a^2b^3)$
$= a^4 - 4a^3b^3 + 6a^2b^6$

9) $(a^4 - 2a^2b^2 + b^4) : (a^2 - b^2) = a^2 - b^2$

10) $[a^2bx^8 - (a^3b - a^5)x^7 - 8a^6x^6 + 7a^7x^5] : (a^2x^2 - a^3x)$
$= bx^6 + a^3x^5 - 7a^4x^4$

11) $(-a^8b^4 + 15a^{11}b^5 - 48a^{14}b^6 - 20a^{17}b^7) : (10a^9b^2 - a^6b)$
$= a^2b^3 - 5a^5b^4 - 2a^8b^5$

12) $(a^8 - 16z^8) : (a^2 - 2z^2) = a^6 + 2a^4z^2 + 4a^2z^4 + 8z^6$

13) $(2a^4 - 13a^3b + 31a^2b^2 - 38ab^3 + 24b^4) :$
$(2a^2 - 3ab + 4b^2) = a^2 - 5ab + 6b^2$

14) $(4c^4 - 9b^2c^2 + 6b^3c - b^4) : (2c^2 - 3bc + b^2)$
$= 2c^2 + 3bc - b^2$

15) $(\frac{3}{4}x^5 - 4x^4 + \frac{77}{8}x^3 - \frac{43}{4}x^2 - \frac{33}{4}x + 27) : (\frac{1}{2}x^2 - x + 3)$
$= \frac{3}{2}x^3 - 5x^2 + \frac{1}{4}x + 9$

16) $(-1 + a^3n^3) : (-1 + an) = 1 + an + a^2n^2$

17) $(3a^4b^{12} - 8a^7b^8 - \frac{51}{2}a^{10}b^6 + \frac{5}{4}a^{10}b^4 + \frac{17}{4}a^{13}b^2)$
$: (\frac{3}{2}a^3b^5 - \frac{1}{4}a^6b) = 2ab^7 - 5a^4b^3 - 17a^7b$

18) $(5a^5b^3c^5 - 22a^4b^3c^6 + 5a^3b^3c^7 + 12a^2b^3c^8 - 7a^2b^2c^8$
$+ 28ab^2c^9) : (a^2bc^2 - 4abc^3) = 5a^3b^2c^3 - 2a^2b^2c^4$
$- 3ab^2c^5 - 7bc^5$

19) $\left(\frac{a^7b^2}{5} - \frac{47a^6b^3}{40} + \frac{9a^5b^4}{2} - 12a^4b^5\right) :$
$\left(\frac{2a^3b^2}{5} - \frac{3a^2b^3}{4} + 6ab^4\right) = \frac{1}{2}a^4 - 2a^3b$

20) $(-2a^{-8}x^5 + 17a^{-4}x^6 - 5x^7 - 24a^4x^8) : (2a^{-3}x^3 - 3ax^4)$
$= -a^{-5}x^2 + 7a^{-1}x^3 + 8a^3x^4$

21) $\left(\frac{a^3c}{b^5} + \frac{a^4c}{b^4} - \frac{7a^5c}{b^3} - \frac{3a^6c}{b^2} + \frac{a^2c^3}{b^2} - \frac{2a^3c^3}{b} - a^4c^3\right)$
$: \left(\frac{a}{b^3} + \frac{3a^2}{b^2} + c^2\right) = \frac{a^2c}{b^2} - \frac{2a^3c}{b} - a^4c$

22) $(a^3d^3 - 3a^2cd^3 + 3ac^2d^3 - c^3d^3 + a^2c^2d^2 - ac^3d^2) :$
$(a^2d^2 - 2acd^2 + c^2d^2 + ac^2d) = ad - cd$

23) $(a^6 + 2a^3z^3 + z^6) : (a^2 - az + z^2) = a^4 + a^3z$
$+ az^3 + z^4$

24) $(\frac{1}{3} - 6z^2 + 27z^4) : (\frac{1}{3} + 2z + 3z^2) = 1 - 6z + 9z^2$

25) $(a^6 - 16a^3x^3 + 64x^6) : (a^2 - 4ax + 4x^2) = a^4 + 4a^3x$
$+ 12a^2x^2 + 16ax^3 + 16x^4$

26) $(a^{3m-2n}b^{2p}c - a^{2m+n-1}b^{1-p}c^n + a^{-n}b^{-1}c^m + a^{3m-n}b^{3p+2}c^n$
$- a^{2m+2n-1}b^3c^{2n-1} + b^{p+1}c^{m+n-1}) : (a^{-n}b^{-p-1} + bc^{n-1})$
$= a^{3m-n}b^{3p+1}c - a^{2m+2n-1}b^2c^n + b^pc^m$

27) $(a^{m+n}b^n - 4a^{m+n-1}b^{2n} - 27a^{m+n-2}b^{3n} + 42a^{m+n-3}b^{4n})$
$: (a^nb^n - 7a^{n-1}b^{2n}) = a^m + 3a^{m-1}b^n - 6a^{m-2}b^{2n}$

28) $(a^n - b^n) : (a - b) = a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots\ldots\ldots$
$+ b^{n-1}$

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c. CASES IN WHICH THE DIVISOR IS NOT AN EXACT NUMBER OF TIMES CONTAINED IN THE DIVIDEND.

1) $\frac{a}{1+x} = a - ax + ax^2 - ax^3 + ax^4 - \ldots\ldots$

2) $\frac{a}{1-x} = a + ax + ax^2 + ax^3 + ax^4 + \ldots\ldots$

3) $\frac{a}{x+1} = \frac{a}{x} - \frac{a}{x^2} + \frac{a}{x^3} - \frac{a}{x^4} + \ldots\ldots$

4) $\frac{a}{x-1} = \frac{a}{x} + \frac{a}{x^2} + \frac{a}{x^3} + \frac{a}{x^4} + \ldots\ldots$

5) $\frac{a+x}{b+x} = \frac{a}{b} - \frac{a-b}{b^2}x + \frac{a-b}{b^3}x^2 - \frac{a-b}{b^4}x^3$
$+ \frac{a-b}{b^5}x^4 \ldots\ldots$

6) $\frac{a-x}{b-x} = \frac{a}{b} + \frac{a-b}{b^2}x + \frac{a-b}{b^3}x^2 + \frac{a-b}{b^4}x^3$
$+ \frac{a-b}{b^5}x^4 \ldots\ldots$

7) $\frac{x+a}{x-b} = 1 + \frac{a+b}{x} + \frac{b(a+b)}{x^2} + \frac{b^2(a+b)}{x^3}$
$+ \frac{b^3(a+b)}{x^4} \ldots\ldots$

4) POWERS OF POWERS.

1) $\left[\left((a^m)^n\right)^p\right]^q = a^{mnpq}$

2) $\left[\left((a^{-m})^{-n}\right)^p\right]^q = a^{mnpq}$

3) $\left[\left((a^{-m})^{-n}\right)^{-p}\right]^{-q} = a^{mnpq}$

4) $\left[\left((a^m)^{-n}\right)^{-p}\right]^{-q} = a^{-mnpq}$

5) $\left[(a^3\,b\,c^2)^5\right]^6 = a^{90}b^{30}c^{60}$

6) $(a^{-2}b^3c^{-5}f^6x^{-1})^{-3} = a^6b^{-9}c^{15}f^{-18}x^3$

7) $(a^mb^{-n}c^pd)^r = a^{mr}b^{-nr}c^{pr}d^r$

8) $(a^{3m-n}f^{2n-1}x^n)^{-3m} = a^{3mn-9mm}f^{3m-6mn}x^{-3mn}$

9) $[a^3\,(a+b)^2]^n = a^{3m}(a+b)^{2m}$

10) $\left(\frac{a^mb^nc^pd^{-q}}{f^ng^{-m}}\right)^{-r} = \frac{a^{-mr}b^{-nr}c^{-pr}d^{qr}}{f^{-nr}g^{mr}}$

11) $\left(\frac{d^4b^5}{c^3df}\right)^4 = \frac{a^{16}b^{20}}{c^{12}d^4f^4}$

12) $\left[\left(\frac{a^2b^3}{c\,d^5}\right)^{-1}\right]^{-2m} = \frac{a^{4m}b^{6m}}{c^{2m}d^{10m}} = \left(\frac{a^4b^6}{c^2d^{10}}\right)^m$

13) $(-a^2)^5 = -a^{10}$

14) $(-b^{-3})^4 = b^{-12}$

15) $\left[\left((-a)^3\right)^4\right]^5 = a^{60}$

16) $[(-a)^{-3}]^{-5} = -a^{15}$

17) $[(-a)^{-4}]^{-6} = a^{24}$

18) $(-a)^{2m} = a^{2m}$

19) $(-a)^{2m+1} = -a^{2m+1}$

20) $\left[\left(-\frac{a}{b}\right)^3\right]^{-4} = \frac{b^{12}}{a^{12}}$

V. ON ROOTS.

Questions to be answered. What is meant by extracting the Root of a number? Are there numbers whereof the root can neither be expressed in integers, nor in vulgar or decimal fractions? How can you prove that there are such roots, which can neither be expressed by whole numbers nor by fractions? What is the name of such roots? What do the words *commensurable* and *incommensurable* denote? Irrational numbers compared with integers or vulgar fractions are necessarily incommensurable; but are they always so when compared to one another? Give examples when they are commensurable with one another? What abbreviations are there in the addition and subtraction of roots; and in what cases? What abbreviations can be made in the multiplication and division of roots? How can the radical exponent be doubled, tripled, &c, without changing the value of the root? When does this become necessary?

1) SQUARE AND CUBE ROOTS OF NUMBERS.

a) SQUARE ROOTS.

1) $\sqrt{256} = 16$

2) $\sqrt{4096} = 64$

3) $\sqrt{61009} = 247$

4) $\sqrt{582169} = 763$

5) $\sqrt{956484} = 978$

6) $\sqrt{57198969} = 7563$

7) $\sqrt{68492176} = 8276$

8) $\sqrt{25836889} = 5083$

9) $\sqrt{236144689} = 15367$

10) $\sqrt{1607448649} = 40093$

11) $\sqrt{780811249} = 27943$

12) $\sqrt{1420913025} = 37695$

13) $\sqrt{285970396644} = 534762$

14) $\sqrt{41605800625} = 203975$

15) $\sqrt{48303584206084} = 6950078$

16) $\sqrt{12088868379025} = 3476905$

17) $\sqrt{5} = 2{,}23606....$

18) $\sqrt{13} = 3{,}60555....$

19) $\sqrt{22} = 4{,}69041....$

20) $\sqrt{96} = 9{,}79795....$

21) $\sqrt{153} = 12{,}36931....$

22) $\sqrt{101} = 10{,}04987....$

23) $\sqrt{7,65}$ = 2,76586....

24) $\sqrt{9,6}$ = 3,09838....

25) $\sqrt{15,2379}$ = 3,90357....

26) $\sqrt{0,056}$ = 0,23664....

27) $\sqrt{0,00789}$ = 0,08882....

28) $\sqrt{0,003}$ = 0,05477....

29) $\sqrt{0,014}$ = 0,11832....

30) $\sqrt{\frac{1}{4}} = \frac{1}{2}$

31) $\sqrt{\frac{9}{16}} = \frac{3}{4}$

32) $\sqrt{\frac{64}{81}} = \frac{8}{9}$

33) $\sqrt{\frac{256}{361}} = \frac{16}{19}$

34) $\sqrt{\frac{7}{4}}$ = 1,32287....

35) $\sqrt{\frac{14}{9}}$ = 1,24721....

36) $\sqrt{11\frac{11}{16}}$ = 3,41869....

37) $\sqrt{7\frac{13}{36}}$ = 2,71313....

38) $\sqrt{8\frac{15}{49}}$ = 2,88203....

39) $\sqrt{\frac{5}{3}}$ = 1,29099....

40) $\sqrt{\frac{7}{8}}$ = 0,93541....

41) $\sqrt{\frac{5}{12}}$ = 0,64549....

42) $\sqrt{\frac{1}{17}}$ = 0,24253....

43) $\sqrt{10\frac{3}{10}}$ = 3,20936....

b. CUBE ROOTS.

1) $\sqrt[3]{12167} = 23$

2) $\sqrt[3]{884736} = 96$

3) $\sqrt[3]{405224} = 74$

4) $\sqrt[3]{2460375} = 135$

5) $\sqrt[3]{11089567} = 223$

6) $\sqrt[3]{1191016} = 106$

7) $\sqrt[3]{17173512} = 258$

8) $\sqrt[3]{49836032} = 368$

9) $\sqrt[3]{40353607} = 343$

10) $\sqrt[3]{64481201} = 401$

11) $\sqrt[3]{8000000} = 200$

12) $\sqrt[3]{74088000} = 420$

13) $\sqrt[3]{318611987} = 683$

14) $\sqrt[3]{340068392} = 698$

15) $\sqrt[3]{6372783864} = 1854$

16) $\sqrt[3]{7256313856} = 1936$

17) $\sqrt[3]{111980168000} = 4820$

18) $\sqrt[3]{115145914625} = 4865$

19) $\sqrt[3]{18970074963} = 2667$

20) $\sqrt[3]{113028882875} = 4835$

21) $\sqrt[3]{8108486729} = 2009$

22) $\sqrt[3]{12} = 2{,}28942....$

23) $\sqrt[3]{82} = 4{,}34448....$

24) $\sqrt[3]{267} = 6{,}43927....$

25) $\sqrt[3]{551} = 8{,}19817....$

26) $\sqrt[3]{687} = 8{,}82373....$

27) $\sqrt[3]{5{,}8} = 1{,}79670....$

28) $\sqrt[3]{102{,}875} = 4{,}68565....$

29) $\sqrt[3]{28{,}25} = 3{,}04559....$

30) $\sqrt[3]{58230{,}605376} = 38{,}76$

31) $\sqrt[3]{\frac{1}{8}} = \frac{1}{2}$

32) $\sqrt[3]{\frac{27}{64}} = \frac{3}{4}$

33) $\sqrt[3]{\frac{343}{512}} = \frac{7}{8}$

34) $\sqrt[3]{\frac{729}{125}} = \frac{9}{5}$

35) $\sqrt[3]{465\frac{31}{64}} = 7\frac{3}{4}$

36) $\sqrt[3]{52034\frac{10}{27}} = 37\frac{1}{3}$

37) $\sqrt[3]{\frac{2}{3}} = 0{,}87358....$

38) $\sqrt[3]{\frac{5}{6}} = 0{,}94103....$

39) $\sqrt[3]{\frac{5}{14}} = 0{,}70949....$

40) $\sqrt[3]{3\frac{4}{5}} = 1{,}56049....$

41) $\sqrt[3]{15\frac{2}{3}} = 2{,}50222....$

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2) ROOTS OF ALGEBRAIC EXPRESSIONS.

a. ROOTS OF SIMPLE ALGEBRAIC EXPRESSIONS.

1) $\sqrt[m]{a^{mn}} = a^n$

2) $\sqrt[m]{a^{-mn}} = a^{-n}$

3) $\sqrt[m]{a^{mn}b^{mp}c^{-mq}d^{-mr}} = a^n b^p c^{-q} d^{-r}$

4) $\sqrt[m]{\frac{a^{mn}b^{mp}}{c^{mq}u^{mr}f^{ms}}} = \frac{a^n b^p}{c^q d^r f^s}$

5) $\sqrt{9a^4b^2f^{-12}g^{-8n}} = 3a^2bf^{-6}g^{-4m}$

6) $\sqrt{\frac{4}{25}c^6x^{2m+4}y^{4p-8}} = \frac{2}{5}c^3x^{m+2}y^{2p-4}$

7) $\sqrt[4]{\frac{a^8b^{20}c^4}{(a+f)^4h^{12}z^{16}}} = \frac{a^2b^5c}{(a+f)h^3z^4}$

8) $\sqrt[3]{\frac{27u^9x^{12}(a^2+x^2)^{-6n}}{8b^{-3n}h^9}} = \frac{3b^nu^3x^4}{2h^3(a^2+x^2)^{2n}}$

9) $\sqrt[5]{\frac{3^{10}a^{-5}(a+b)^5(2+x)^{-10}}{32c^5d^{-15}}} = \frac{9d^3(a+b)}{2ac(2+x)^2}$

10) $\sqrt[9]{\left(2^{36}a^{45}b^9 \times \frac{(a+b)^{27}}{a^9}\right)} = 2^4a^4b(a+b)^3$

11) $\sqrt{\left(\frac{a^2b^2c^2}{d^4} \times \frac{1}{a^6b^8f^4}\right)} = \frac{c}{a^2b^3d^2f^2}$

---

b. SQUARE ROOTS OF COMPOUND ALGEBRAIC EXPRESSIONS.

1) $\surd(a^2 + 2ab + b^2) = a + b$

2) $\surd(a^2 - 2\,ab + b^2) = a - b$

3) $\surd\left(a^2 - ab + \frac{b^2}{4}\right) = a - \frac{b}{2}$

4) $\surd(x^2 + 2x + 1) = x + 1$

5) $\surd(f^6 + 6f^3x^4 + 9x^8) = f^3 + 3x^4$

6) $\surd\left(\frac{9a^8}{4} + 2a^4n^3 + \frac{4n^6}{9}\right) = \frac{3a^4}{2} + \frac{2n^3}{3}$

7) $\surd(\frac{25}{4}a^2b^2 - \frac{5}{3}abc^2 + \frac{1}{9}c^4) = \frac{5}{2}ab - \frac{1}{3}c^2$

8) $\surd(x^4 - ax^3 + \frac{1}{4}a^2x^2) = x^2 - \frac{1}{2}ax$

9) $\surd(a^{2m} + 2a^mx^n + x^{2n}) = a^m + x^n$

10) $\surd(a^{2m} - 4a^{m+n} + 4a^{2n}) = a^m - 2a^n$

11) $\surd\left(\frac{a^2}{b^2} - \frac{4a}{3c} + \frac{4b^2}{9c^2}\right) = \frac{a}{b} - \frac{2b}{3c}$

12) $\surd(a^2 + 2ab + 2ac + b^2 + 2bc + c^2) = a + b + c$

13) $\surd\left(9x^2 - 30ax - 3a^2x + 25a^2 + 5a^3 + \frac{a^4}{4}\right)$
$= 3x - 5a - \frac{a^2}{2}$

14) $\surd(4x^4 + 8ax^3 + 4a^2x^2 + 16b^2x^2 + 16ab^2x + 16b^4)$
$= 2x^2 + 2ax + 4b^2$

15) $\surd(9a^2 - 6ab + 30ac + 6ad + b^2 - 10bc - 2bd$
$+ 25c^2 + 10cd + d^2) = 3a - b + 5c + d$

16) $\surd(\frac{9}{4} + 6x - 17x^2 - 28x^3 + 49x^4) = \frac{3}{2} + 2x - 7x^2$

17) $\sqrt{}(9x^4 - 3ax^3 + 6bx^3 + \frac{a^2x^2}{4} - abx^2 + b^2x^2)$

$= 3x^2 - \frac{ax}{2} + bx$

18) $\sqrt{}(\frac{4}{9}a^2x^4 - \frac{4}{3}abx^3z + \frac{8}{3}a^2bx^2z^2 + b^2x^2z^2 - 4ab^2xz^3$

$+ 4a^2b^2z^4) = \frac{2}{3}ax^2 - bxz + 2abz^2$

19) $\sqrt{\frac{a^2 - 2ab + b^2}{x^4 + 4ax^2 + 4a^2}} = \frac{a - b}{x^2 + 2a}$

20) $\sqrt{\frac{a^2x^2 + 2ab^2x^3 + b^4x^4}{a^{2m} + 2a^mx^n + x^{2n}}} = \frac{ax + b^2x^2}{a^m + x^n}$

21) $\sqrt{}(a^{2m}x^{2n} + 10ca^{2m-2}x^{2n+1} - 6a^{m+1}x^{n-1}$

$+ 25c^2a^{2m-4}x^{2n+2} - 30ca^{m-1}x^n + \frac{9a^2}{x^2})$

$= a^mx^n + 5ca^{m-2}x^{n+1} - \frac{3a}{x}$

22) $\sqrt{}\left(\frac{9a^{2m-2}c^2}{4d^{6p}} - \frac{3a^{m+n-1}b^{2n-1}c}{d^{3p-3}} - \frac{2^8a^{m-1}b^xc}{d^{3p}}\right.$

$\left.+ a^{2n}b^{4n-2}d^6 + \frac{2^9}{3}a^nb^{x+2n-1}d^3 + \frac{2^{16}b^{2x}}{9}\right)$

$= \frac{3a^{m-1}c}{2d^{3p}} - a^nb^{2n-1}d^3 - \frac{2^8b^x}{3}$

---

c. CUBE ROOTS OF COMPOUND ALGEBRAIC EXPRESSIONS.

1) $\sqrt[3]{}(a^3 + 3a^2b + 3ab^2 + b^3) = a + b$

2) $\sqrt[3]{}(a^3 - 3a^2b + 3ab^2 - b^3) = a - b$

3) $\sqrt[3]{}(x^3 + 6x^2 + 12x + 8) = x + 2$

4) $\sqrt[3]{(8a^3 - 84a^2x + 294ax^2 - 343x^3)} = 2a - 7x$

5) $\sqrt[3]{(x^6 - 6cx^5 + 12c^2x^4 - 8c^3x^3)} = x^2 - 2cx$

6) $\sqrt[3]{(a^{3m} - 6a^{2m+1}x^n + 12a^{m+2}x^{2n} - 8a^3x^{3n})} = a^m - 2ax^n$

7) $\sqrt[3]{(8 - 12x^{3n-1} + 6x^{6n-2} - x^{9n-3})} = 2 - x^{3n-1}$

8) $\sqrt[3]{\left(\frac{a^3c^3}{b^3}x^6 - \frac{3a^2c}{b}x^5 + \frac{3ab}{c}x^4 - \frac{b^3}{c^3}x^3\right)} = \frac{ac}{b}x^2 - \frac{b}{c}x$

9) $\sqrt[3]{\left(b^3 + \frac{3a^2b^2}{2c^2}x^{-2} + \frac{3a^4b}{4c^4}x^{-4} + \frac{a^6}{8c^6}x^{-6}\right)} = b + \frac{a^2}{2c^2}x^{-2} = b + \frac{a^2}{2c^2x^2}$

10) $\sqrt[3]{(a^3 + 3a^2b + 3a^2c + 3ab^2 + 6\,abc + 3ac^2 + b^3 + 3b^2c + 3bc^2 + c^3)} = a + b + c$

11) $\sqrt[3]{(27z^6 - 54az^5 + 63a^2z^4 - 44a^3z^3 + 21a^4z^2 - 6a^5z + a^6)} = 3z^2 - 2az + a^2$

12) $\sqrt[3]{\left(\frac{a^3y^3}{b^6c^3} + \frac{3a^2cy^4}{b^4d} - \frac{3a^3y^2}{b^4c^2} + \frac{3ac^5y^5}{b^2d^2} - \frac{6a^2c^2y^3}{b^2d} + \frac{3a^3y}{b^2c} + \frac{c^9y^6}{d^3} - \frac{3ac^6y^4}{d^2} + \frac{3a^2c^3y^2}{d} - a^3\right)} = \frac{ay}{b^2c} + \frac{c^3y^2}{d} - a$

13) $\sqrt[3]{(8x^6 + 48cx^5 + 60c^2x^4 - 80c^3x^3 - 90c^4x^2 + 108c^5x - 27c^6)} = 2x^2 + 4cx - 3c^2$

14) $\sqrt[3]{[(a + b)^{6m}x^3 + 6ca^p(a + b)^{4m}x^2 + 12c^2a^{2p} \times (a + b)^{2m}x + 8c^3a^{3p}]} = (a + b)^{2m}x + 2ca^p$

d. SQUARE AND CUBE ROOTS OF IMPERFECT SQUARES AND CUBES.

1) $\sqrt{(a^2 - x^2)} = a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5}$
$- \frac{5x^8}{128a^7} - \ldots$

2) $\sqrt{(a^2 + x^2)} = a + \frac{x^2}{2a} - \frac{x4}{8a^3} + \frac{x^6}{16a^5}$
$- \frac{5x^8}{128a^7} + \ldots$

3) $\sqrt{(1 - x)} = 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128} - \ldots$

4) $\sqrt{(1 + x)} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \ldots$

5) $\sqrt[3]{(a^3 - x^3)} = a - \frac{x^3}{3a^2} - \frac{x^6}{9a^5} - \frac{5x^9}{81a^8}$
$- \frac{10x^{12}}{243a^{11}} - \ldots$

6) $\sqrt[3]{(a^3 + x^3)} = a + \frac{x^3}{3a^2} - \frac{x^6}{9a^5} + \frac{5x^9}{81a^8}$
$- \frac{10x^{12}}{243a^{11}} + \ldots$

7) $\sqrt[3]{(1 - x)} = 1 - \frac{x}{3} - \frac{x^2}{9} - \frac{5x^3}{81} - \frac{10x^4}{243} - \ldots$

8) $\sqrt[3]{(1 + x)} = 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81} - \frac{10x^4}{243} + \ldots$

The teacher ought to show his pupils the use of these series in the extraction of square and cube roots. For this purpose he may substitute in the formulæ (1), (2), (5), (6), for *a*, the numbers 2, 3, 4, 5, and for *x* constantly 1; and in the formulæ (3), (4), (7), (8), for *x* any of the fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, &c.

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3) ADDITION, SUBTRACTION, MULTIPLICATION AND DIVISION OF ROOTS.

a. ADDITION AND SUBTRACTION.

1) $b\sqrt[m]{a} + c\sqrt[m]{a} - d\sqrt[m]{a} = (b + c - d)\sqrt[m]{a}$

2) $3\sqrt[6]{5} + 17\sqrt[6]{5} - 12\sqrt[6]{5} - 7\sqrt[6]{5} = \sqrt[6]{5}$

3) $6\sqrt{2} - 5\sqrt{2} + \tfrac{3}{4}\sqrt{2} - \tfrac{1}{2}\sqrt{2} = \tfrac{5}{4}\sqrt{2}$

4) $6\sqrt[4]{\tfrac{3}{2}} - 2\sqrt[4]{\tfrac{3}{2}} + a\sqrt[4]{\tfrac{3}{2}} - \frac{2b}{c}\sqrt[4]{\tfrac{3}{2}} = \left(4 + a - \frac{2b}{c}\right)\sqrt[4]{\tfrac{3}{2}}$

5) $5\sqrt[7]{9} - 2\sqrt[5]{14} + \sqrt[3]{2} - 5\sqrt[5]{14} - 2\sqrt[7]{9} = 3\sqrt[7]{9}$
$- 7\sqrt[5]{14} + \sqrt[3]{2}$

6) For Addition.

$$\left\{\begin{array}{l}
10\sqrt{2} + 5\sqrt[7]{8} - 7\sqrt[m]{5} + 2\sqrt[3]{a} \\
5\sqrt{2} + \sqrt[7]{8} + 4\sqrt[m]{5} - 3\sqrt[3]{a} \\
- 3\sqrt{2} - 9\sqrt[7]{8} - 3\sqrt[m]{5} + \sqrt[3]{a} + \sqrt{ab} \\
\hline
12\sqrt{2} - 3\sqrt[7]{8} - 6\sqrt[m]{5} + \sqrt{ab}
\end{array}\right.$$

7) For Addition.

$$\left\{\begin{array}{l}
13\sqrt[5]{12a^2bc} + 17\sqrt[m]{3} - 5\sqrt[n]{6} \\
7\sqrt[5]{12a^2bc} + 2\sqrt[n]{6} + 3\sqrt[m]{3} - 2a\sqrt{c} + \tfrac{1}{2}\sqrt[7]{9a} \\
-20\sqrt[5]{12a^2bc} + 9\sqrt[5]{12c^2bc} + \sqrt{c} - 3\sqrt[7]{9a} \\
\hline
20\sqrt[m]{3} - 3\sqrt[n]{6} + 9\sqrt[5]{12a^2bc} - (2a - 1)\sqrt{c} - \tfrac{5}{2}\sqrt[7]{9a}
\end{array}\right.$$

8) For Subt.

$$\left\{\begin{array}{l}
18\sqrt{7} - 5\sqrt{6} + 10\sqrt[4]{11} - 3\sqrt[3]{13} \\
6\sqrt{7} - 2\sqrt{6} + \tfrac{17}{2}\sqrt[4]{11} + 2\sqrt[3]{13} \\
\hline
12\sqrt{7} - 3\sqrt{6} + \tfrac{3}{2}\sqrt[4]{11} - 5\sqrt[3]{13}
\end{array}\right.$$

9) For Subt. $$\left\{\begin{array}{l} 16\sqrt[4]{6a\,b} - \sqrt[5]{9c^3} + 3\sqrt[m]{7a} - \sqrt{10} \\ -\,8\sqrt[5]{9c^3} - 5\sqrt[m]{7a} + 3\sqrt[4]{6ab} + 2\sqrt[4]{10} \\ \hline 13\sqrt[4]{6ab} + 7\sqrt[5]{9c^3} + 8\sqrt[m]{7a} - \sqrt{10} - 2\sqrt[4]{10} \end{array}\right.$$

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b. ABBREVIATIONS AND TRANSFORMATIONS.

1) $\sqrt{24} + \sqrt{54} - \sqrt{6} = 4\sqrt{6}$

2) $2\sqrt{8} - 7\sqrt{18} + 5\sqrt{72} - \sqrt{50} = 8\sqrt{2}$

3) $\sqrt{12} + 2\sqrt{27} + 3\sqrt{75} - 9\sqrt{48} = -13\sqrt{3}$

4) $8\sqrt{\frac{3}{4}} - \frac{1}{2}\sqrt{12} + 4\sqrt{27} - 2\sqrt{\frac{3}{16}} = \frac{29}{2}\sqrt{3} = \frac{29\sqrt{3}}{2}$

5) $2\sqrt{\frac{5}{3}} + \sqrt{60} - \sqrt{15} + \sqrt{\frac{3}{5}} = \frac{28}{15}\sqrt{15}$

6) $7\sqrt[3]{54} + 3\sqrt[3]{16} + \sqrt[3]{2} - 5\sqrt[3]{128} = 8\sqrt[3]{2}$

7) $\sqrt[3]{81} - 2\sqrt[3]{24} + \sqrt{28} + 2\sqrt{63} = 8\sqrt{7} - \sqrt[3]{3}$

8) $\sqrt[4]{32} + 2\sqrt[3]{40} = 2\sqrt[4]{2} + 4\sqrt[3]{5}$

9) $3\sqrt{5} - 2\sqrt{2} + 3\sqrt{6} = \sqrt{45} - \sqrt{8} + \sqrt{54}$

10) $5\sqrt[3]{7} + 3\sqrt{2} + 2\sqrt[4]{3} = \sqrt[3]{875} + \sqrt{18} + \sqrt[4]{48}$

11) $4\sqrt[5]{\frac{1}{2}} + 3\sqrt[3]{2} - 5\sqrt{\frac{1}{3}} = \sqrt[5]{512} + \sqrt[3]{54} - \sqrt{\frac{25}{3}}$

12) $\sqrt{45c^3} - \sqrt{80c^3} + \sqrt{5a^2c} = (a-c)\sqrt{5c}$

13) $\sqrt{18a^5b^3} + \sqrt{50a^3b^3} = (3a^2b + 5ab)\sqrt{2ab}$

14) $\sqrt[3]{16a^3b} + \sqrt{4a^2b} - \sqrt{a^2b} - \sqrt[3]{54a^3b} = a\sqrt{b} - a\sqrt[3]{2b}$

15) $\sqrt[4]{2^{14}a^{13}b^5c} - \sqrt[4]{4\cdot5^4a^5b^9c^5} + \sqrt[4]{4\cdot6^4ab^5c} =$

$(8a^3b - 5ab^2c + 6b)\sqrt[4]{4abc}$

16) $\sqrt{\frac{a^4c}{b^3}} + \sqrt{\frac{a^2c^3}{bd^2}} - \sqrt{\frac{a^2cd^2}{be^2}} = \left(\frac{a^2}{b} + \frac{ac}{d} - \frac{ad}{e}\right)\sqrt{\frac{c}{b}}$

17) $\sqrt[3]{\frac{27a^5x}{2b}} - \sqrt[3]{\frac{a^2x}{2b}} = (3a - 1)\sqrt[3]{\frac{a^2x}{2b}}$

18) $3b^2\sqrt{a^3c} + \frac{2}{c}\sqrt{a^5c^3} - c^4\sqrt{\frac{ac}{b^2}} = \left(3ab^2 + 2a^2 - \frac{c^4}{b}\right)\sqrt{ac}$

19) $5a\sqrt[3]{\frac{a^2}{b}} + b\sqrt[3]{\frac{b^2c^3}{a}} = \left(\frac{5a}{b} + \frac{bc}{a}\right)\sqrt[3]{a^2b^2}$

20) $\sqrt[3]{54a^{m+6}b^3} - \sqrt[3]{16a^{m-3}b^6} + \sqrt[3]{2a^{4m+9}} + \sqrt[3]{2c^3a^m}$
$= (3a^2b - \frac{2b^2}{a} + a^{m+3} + c)\sqrt[3]{2a^m}$

21) $\sqrt[m]{2^ma^{mp+3}b^{mn+5}} + \sqrt[m]{3^ma^{2m-mn+3}b^{m+5}} - \sqrt[m]{a^3b^5c^{2m}}$
$= (2a^pb^n + 3a^{2-n}b - c^2)\sqrt[m]{a^3b^5}$

22) $\sqrt[6]{\frac{3 \cdot 2^3c^3f^4}{d^4g}} + \sqrt[6]{\frac{2^3g^{11}}{3^5c^3d^4f^2}} = \left(\frac{f}{d} + \frac{g^2}{3cd}\right)\sqrt[6]{\frac{3 \cdot 2^3c^3d^2}{f^2g}}$

23) $\sqrt[2n]{\frac{a^{6n+4}b^{2n-3}c^{2mn}}{d^{9n+5}f^2g^{p+2n-1}}} = \frac{a^3bc^m}{d^4g}\sqrt[2n]{\frac{a^4}{d^{n+5}f^2g^{p-1}b^3}}$

24) $\sqrt{(a^2c + a^2d)} = a\sqrt{(c + d)}$

25) $\sqrt[3]{(a^{6m}b - a^{7m}f^2)} = a^{2m}\sqrt[3]{(b - a^mf^2)}$

26) $\sqrt[7]{\left(\frac{a^{10}}{b^6c^6} + \frac{a^8}{b^5c^5}\right)} = \frac{a}{bc}\sqrt[7]{(a^3bc + ab^2c^2)}$

27) $\sqrt{\left(\frac{a^3b^2}{cd^2} - \frac{2a^2b^3}{c^2d}\right)} = \frac{ab}{cd}\sqrt{(ac - 2bd)}$

28) $x\sqrt[3]{\left(\frac{8a^4}{27b^3} + \frac{16a^3}{27b^2}\right)} = \frac{2ax}{3b}\sqrt[3]{(a + 2b)}$

29) $\sqrt{(3a^2c + 6abc + 3b^2c)} = (a + b)\sqrt{3c}$

30) $\sqrt{(4a^5b^2 - 20a^3b^3 + 25ab^4)} = (2a^2 - 5b)\sqrt{ab^2}$

31) $\sqrt{(2ax^2 - 4ax + 2a)} = (x - 1)\sqrt{2a}$

32) $\sqrt{\frac{a^3b - 4a^2b^2 + 4ab^3}{c^2d^2}} = \frac{a - 2b}{cd}\sqrt{ab}$

33) $\sqrt{\frac{a^2x - 2ax^2 + x^3}{a^2 + 2ax + x^2}} = \frac{a - x}{a + x}\sqrt{x}$

34) $\sqrt{\frac{ac}{a^2bd - 2ab^2d + b^3d}} = \frac{1}{a - b}\sqrt{\frac{ac}{bd}}$

35) $\sqrt{\frac{x^3 + 2x^2 + x}{a^3 + a^2b}} = \frac{x + 1}{a}\sqrt{\frac{x}{a + b}}$

36) $\sqrt{\frac{a^3 - a^2x - ax^2 + x^3}{b^5c^3d}} = \frac{a - x}{b^2c}\sqrt{\frac{a + x}{bcd}}$

37) $\frac{a - b}{a + b}\sqrt{\frac{ac}{a^2 - 2ab + b^2}} = \frac{\sqrt{ac}}{a + b}$

38) $\frac{a + b}{a - b}\sqrt{\frac{a - b}{a + b}} = \sqrt{\frac{a + b}{a - b}}$

39) $(x + 1)\sqrt{\frac{f^2g}{x^2 - 1}} = \sqrt{\frac{(x + 1)^2f^2g}{(x + 1)(x - 1)}} = \sqrt{\frac{(x + 1)f^2g}{x - 1}}$

40) $\sqrt[m]{\sqrt[n]{\sqrt[p]{\sqrt[q]{a}}}} = \sqrt[mnpq]{a}$

41) $\sqrt[3]{(2\sqrt{5})} = \sqrt[6]{20}$

42) $\sqrt[m]{(a\sqrt[n]{b})} = \sqrt[mn]{a^nb}$

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c. MULTIPLICATION.

1) $\sqrt[m]{a} \times \sqrt[m]{b} \times \sqrt[m]{c} = \sqrt[m]{abc}$

2) $a\sqrt[n]{x} \times b\sqrt[n]{y} \times c\sqrt[n]{z} = abc\sqrt[n]{xyz}$

3) $\sqrt[3]{4} \times 7\sqrt[3]{6} \times \frac{1}{2}\sqrt[3]{5} = \frac{7}{2}\sqrt[3]{120}$

4) $4 \times 2\sqrt[6]{3} \times \sqrt[6]{72} = 8\sqrt{6}$

5) $5\sqrt{3} \times 7\sqrt{\tfrac{8}{3}} \times \sqrt{2} = 140$

6) $c\sqrt{a} \times d\sqrt{a} = acd$

7) $\sqrt[m]{a} \times \sqrt[n]{b} = \sqrt[mn]{a^n} \times \sqrt[mn]{b^m} = \sqrt[mn]{a^n b^m}$

8) $\sqrt{2} \times \sqrt[3]{3} \times \sqrt[4]{5} = \sqrt[12]{648000}$

9) $\sqrt[3]{2} \times \sqrt[6]{\tfrac{1}{3}} \times \sqrt[8]{3} = \sqrt[24]{\tfrac{256}{3}}$

10) $\sqrt[5]{4} \times \sqrt[10]{3} \times \sqrt[15]{6} = \sqrt[30]{3981312}$

11) $\sqrt[7]{\tfrac{4}{3}} \times \sqrt[3]{\tfrac{1}{2}} \times \sqrt[14]{6} = \sqrt[42]{\tfrac{2}{27}}$

12) $a\sqrt[m]{x} \times b\sqrt[n]{y} \times c\sqrt[p]{z} = abc\sqrt[mnp]{x^{np} y^{mp} z^{mn}}$

13) $\sqrt[12]{\dfrac{a}{bc}} \times \sqrt[8]{\dfrac{a^m}{b}} = \sqrt[24]{\dfrac{a^{3m+2}}{b^5 c^2}}$

14) $\dfrac{ac}{b^3 d^5}\sqrt[3]{\dfrac{bcd}{e}} \times \sqrt[6]{\dfrac{b^{10} d^7 e}{a^2 c^5}} = \dfrac{1}{bd}\sqrt[6]{\dfrac{a^4 c^3}{d^3 e}}$

15) $(\sqrt{5} + 2\sqrt{7} + 3\sqrt{10}) \times 2\sqrt{5} = 10 + 4\sqrt{35} + 6\sqrt{50}$

16) $(\sqrt{6} + \sqrt[3]{2} - 2\sqrt[4]{5}) \times \sqrt{3} = \sqrt{18} + \sqrt[6]{108} - 2\sqrt[4]{45}$

17) $(3 + \sqrt{5}) \times (2 - \sqrt{5}) = 1 - \sqrt{5}$

18) $(7 + 2\sqrt{6}) \times (9 - 5\sqrt{6}) = 3 - 17\sqrt{6}$

19) $(9 - 7\sqrt{13}) \times (5 - 6\sqrt{13}) = 591 - 89\sqrt{13}$

20) $(6 + 12\sqrt{7}) \times (3 - 5\sqrt{7}) = 6\sqrt{7} - 402$

21) $(9\sqrt{12} + 3) \times (5\sqrt{12} + 8) = 564 + 87\sqrt{12}$

22) $(13 - \sqrt{5}) \times (7 + 3\sqrt{5}) = 76 + 32\sqrt{5}$

23) $(\tfrac{3}{2} + \tfrac{5}{2}\sqrt{\tfrac{1}{2}}) \times (\tfrac{1}{2} - 7\sqrt{\tfrac{1}{2}}) = -8 - \tfrac{37}{4}\sqrt{\tfrac{1}{2}}$

24) $(-5 - \sqrt{\tfrac{3}{4}}) \times (-5 + \sqrt{\tfrac{3}{4}}) = 24\tfrac{1}{4}$

25) $(9 + 2\sqrt{10}) \times (9 - 2\sqrt{10}) = 41$

26) $(\sqrt{2}+\sqrt{3}) \times (2\sqrt{2}-\sqrt{3}) = 1+\sqrt{6}$

27) $(5\sqrt{14}+3\sqrt{5}) \times (7\sqrt{14}-2\sqrt{5}) = 460+11\sqrt{70}$

28) $(2\sqrt{7}-5\sqrt{6}) \times \left(\frac{3\sqrt{7}}{2}-2\sqrt{6}\right) = 81-\frac{23}{2}\sqrt{42}$

29) $(4\sqrt{\frac{7}{3}}+5\sqrt{\frac{1}{2}}) \times (\sqrt{\frac{7}{3}}+2\sqrt{\frac{1}{2}}) = \frac{43}{3}+13\sqrt{\frac{7}{6}}$

30) $(\sqrt{2}+\sqrt{3})^3 = 11\sqrt{2}+9\sqrt{3}$

31) $(\sqrt{7}-\sqrt{3}) \times (\sqrt{5}-\sqrt{2}) = \sqrt{35}-\sqrt{15}-\sqrt{14}$
$+\sqrt{6}$

32) $(5-8\sqrt{7}) \times (9+10\sqrt{3}) = 45-72\sqrt{7}+50\sqrt{3}$
$-80\sqrt{21}$

33) $(7\sqrt{6}+2\sqrt{3}) \times (\sqrt{5}+\sqrt{6}) = 7\sqrt{30}+2\sqrt{15}$
$+42+2\sqrt{18}$

34) $(3\sqrt{\frac{1}{2}}-\sqrt{\frac{1}{3}}) \times (5\sqrt{\frac{5}{6}}+\sqrt{\frac{2}{7}}) = 15\sqrt{\frac{5}{12}}-5\sqrt{\frac{5}{18}}$
$+3\sqrt{\frac{1}{7}}-\sqrt{\frac{2}{21}}$

35) $(5\sqrt{3}-7\sqrt{6}) \times (2\sqrt{8}-3) = 41\sqrt{6}-71\sqrt{3}$

36) $(2\sqrt{6}-3\sqrt{5}) \times (4\sqrt{3}-\sqrt{10}) = 39\sqrt{2}-16\sqrt{15}$

37) $(\sqrt{12}-2\sqrt{7}) \times (2+\sqrt{21}) = 2\sqrt{7}-10\sqrt{3}$

38) $(3\sqrt{5}+2\sqrt{6}-2) \times (2\sqrt{5}+18\sqrt{6}) = 246+$
$58\sqrt{30}-4\sqrt{5}-36\sqrt{6}$

39) $(2\sqrt{8}+3\sqrt{5}-7\sqrt{2}) \times (\sqrt{72}-5\sqrt{20}-2\sqrt{2})$
$=-174+42\sqrt{10}$

40) $(2\sqrt{5}+3\sqrt{2}-8\sqrt{6}) \times (2+5\sqrt{2}-3\sqrt{12})$
$=30+4\sqrt{5}+150\sqrt{2}-34\sqrt{6}+10\sqrt{10}$
$-40\sqrt{12}-6\sqrt{60}$

41) $(3\sqrt{2}+2\sqrt{5}+\sqrt{7}) \times (\sqrt{6}+5\sqrt{3}+\sqrt{10})$
$=3\sqrt{12}+2\sqrt{30}+\sqrt{42}+15\sqrt{6}+10\sqrt{15}$
$+5\sqrt{21}+3\sqrt{20}+2\sqrt{50}+\sqrt{70}$

42) $(\sqrt[3]{5} - 2\sqrt[3]{6}) \times (3\sqrt[3]{4} - \sqrt[3]{36}) = 12 + 3\sqrt[3]{20}$
$- 6\sqrt[3]{24} - \sqrt[3]{180}$

43) $(5\sqrt[3]{4} - 2\sqrt[3]{16}) \times (2\sqrt[3]{2} - 3\sqrt[3]{4}) = 44 - 4\sqrt[3]{32}$
$- 15\sqrt[3]{16}$

44) $(2\sqrt{3} + \sqrt[3]{2}) \times (2 + \sqrt[3]{9}) = 4\sqrt{3} + 2\sqrt[3]{2} + \sqrt[3]{18}$
$+ 6\sqrt[6]{3}$

45) $(5 + \sqrt[3]{4} + 2\sqrt[4]{5}) \times (\sqrt{6} + \sqrt{5}) = 5\sqrt{6} + 5\sqrt{5}$
$+ 2\sqrt[4]{125} + 2\sqrt[4]{180} + 2\sqrt[6]{54} + \sqrt[6]{2000}$

46) $(a + \sqrt{b}) \times (a - \sqrt{b}) = a^2 - b$

47) $(\sqrt{a} + \sqrt{b}) \times (\sqrt{a} - \sqrt{b}) = a - b$

48) $(c\sqrt{a} + d\sqrt{b}) \times (c\sqrt{a} - d\sqrt{b} = ac^2 - bd^2$

49) $(a + \sqrt{x}) \times (b + \sqrt{y}) = ab + a\sqrt{y} + b\sqrt{x} + \sqrt{xy}$

50) $\left(\sqrt{\frac{ad^2}{c^3}} + \sqrt{\frac{a^2}{b}}\right) \times (\sqrt{ac} + \sqrt{b^3}) = \frac{ad}{c} + ab$
$+\left(a + \frac{b^2d}{c^2}\right)\sqrt{\frac{ac}{b}}$

51) $\left(\sqrt{\frac{ac^2}{a+b}} + \sqrt{\frac{1}{b}}\right) \times \left(\frac{c}{d}\sqrt{(a+b)\,a} - \sqrt{\frac{b^5}{c^2}}\right) =$
$\frac{ac^2}{d} - \frac{b^2}{c} + \left(\frac{c}{bd} - \frac{b^2}{a+b}\right)\sqrt{(a+b)ab}$

52) $(\sqrt{a} + c\sqrt[3]{b}) \times (\sqrt{a} - c\sqrt[3]{b}) = a - c^2\sqrt[3]{b^2}$

53) $(2\sqrt{a} + 3c\sqrt[3]{b}) \times (\sqrt{a} + 4\sqrt[3]{b}) = 2a + 12c\sqrt[3]{b^2}$
$+ (3c + 8)\sqrt[6]{a^3b^2}$

54) $(c\sqrt[4]{a} + d\sqrt[4]{b}) \times (f\sqrt[4]{a} + g\sqrt[4]{b}) = cf\sqrt{a} + dg\sqrt{b}$
$+ (df + cg)\sqrt[4]{ab}$

55) $(\sqrt[4]{a}+\sqrt[4]{b}+\sqrt[4]{c})^2=\sqrt{a}+\sqrt{b}+\sqrt{c}+2\sqrt[4]{ab}$
$+2\sqrt[4]{ac}+2\sqrt[4]{bc}$

56) $\sqrt{(a+\sqrt{b})}\times\sqrt{(c+\sqrt{d})}=\sqrt{(ac+c\sqrt{b}+a\sqrt{d}}$
$+\sqrt{bd})$

57) $\sqrt[n]{(a+\sqrt{b})}\times\sqrt[n]{(a-\sqrt{b})}=\sqrt[n]{(a^2-b)}$

58) $\sqrt[m]{(a+\sqrt[n]{b})}\times\sqrt[m]{(c+\sqrt[p]{d})}=\sqrt[m]{(ac+c\sqrt[n]{b}+a\sqrt[p]{d}}$
$+\sqrt[np]{b^pd^n})$

59) $\sqrt[4]{(5+2\sqrt{6})}\times\sqrt{(3+\sqrt{6})}=\sqrt[4]{(147+60\sqrt{6})}$

60) $3\sqrt[n]{(2+4\sqrt[3]{3})}\times4\sqrt[n]{(6+2\sqrt[3]{9})}=12\sqrt[n]{(36+4\sqrt[3]{9}}$
$+24\sqrt[3]{3})$

61) $5\sqrt{2}\times3\sqrt{(4+6\sqrt{2})}=30\sqrt{(2+3\sqrt{2})}$

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d. DIVISION.

1) $\sqrt[m]{a}:\sqrt[m]{b}=\sqrt[m]{\frac{a}{b}}$

2) $c\sqrt[m]{a}:d\sqrt[m]{b}=\frac{c}{d}\sqrt[m]{\frac{a}{b}}$

3) $a:\sqrt[m]{b}+\sqrt[m]{\frac{a^m}{b}}$

4) $a:\sqrt{a}=\sqrt{a}$

5) $2ab^2c^3:4\sqrt[3]{a^3bc^5d}=\frac{1}{2}\sqrt[3]{\frac{b^5c^4}{d}}$

6) $\sqrt[5]{ab^{n-1}c^2}:\sqrt[5]{\frac{a^3b^2}{dc^{n-1}}}=\sqrt[5]{\frac{b^{n-3}c^{n+1}d}{a^2}}$

7) $\sqrt[n]{\frac{f^3g^2}{dx^5}}:\sqrt[n]{\frac{fg}{dx}}=\sqrt[n]{\frac{f^2g}{x^4}}$

8) $\sqrt[3]{a^2bc} : \sqrt[5]{ab^2c^3} = \sqrt[15]{\frac{a^7}{bc^4}}$

9) $\sqrt[4]{\frac{a}{b}} : \sqrt{\frac{a}{b}} = \sqrt[4]{\frac{b}{a}}$

10) $4\sqrt[3]{12} : 2\sqrt{3} = 2\sqrt[6]{\frac{16}{3}}$

11) $\sqrt[5]{64} : 2 = \sqrt[5]{2}$

12) $\sqrt[2]{\frac{a^mb}{c^2d}} : \sqrt[3n]{\frac{a^{m-1}c^3}{d^5}} = \sqrt[6n]{\frac{a^{m+2}b^3d^7}{c^{12}}}$

13) $c\sqrt{(a^2 - x^2)} : \sqrt{(a + x)} = c\sqrt{(a - x)}$

14) $\sqrt{(ab^2 - b^2c)} : \sqrt{(a - c)} = b$

15) $\sqrt{(a^2 - z^2)} : (a - z) = \sqrt{\frac{a + z}{a - z}}$

16) $(\sqrt{72} + \sqrt{32} - 4) : \sqrt{8} = 5 - \sqrt{2}$

17) $(\sqrt{6} + 4\sqrt{18} - 3 - 8\sqrt{2}) : \sqrt{3} = \sqrt{2} + 4\sqrt{6}$
$- \sqrt{3} - 8\sqrt{\tfrac{2}{3}}$

18) $(3\sqrt{15} - \sqrt{20} + \sqrt{10} - 7) : 2\sqrt{5} = \tfrac{3}{2}\sqrt{3} - 1$
$+ \tfrac{1}{2}\sqrt{2} - \tfrac{1}{2}\sqrt{\tfrac{49}{5}}$

19) $(2\sqrt{32} + 3\sqrt{2} + 4) : 4\sqrt{8} = \tfrac{11}{8} + \tfrac{1}{4}\sqrt{2}$

20) $(6 + 2\sqrt{3} - \sqrt[3]{18}) : \sqrt{6} = \sqrt{6} + \sqrt{2} - \sqrt[6]{\tfrac{3}{2}}$

21) $(\sqrt{8} + \sqrt[3]{12} + \sqrt[4]{2}) : 2\sqrt{2} = 1 + \frac{\sqrt[6]{18}}{2} + \frac{\sqrt[4]{8}}{4}$

22) $1 : (\sqrt{3} + 2) = 2 - \sqrt{3}$

23) $3 : (1 + \sqrt{2}) = 3\sqrt{2} - 3$

24) $12 : (5 - \sqrt{21}) = 15 + 3\sqrt{21}$

25) $7 : (\sqrt{8} - 2) = \tfrac{7}{2}(\sqrt{2} + 1)$

26) $\sqrt{3} : (2\sqrt{5} - 3\sqrt{2}) = \sqrt{15} + \tfrac{3}{2}\sqrt{6}$

27) $\frac{3}{4}\sqrt{\frac{5}{6}} : (\sqrt{\frac{1}{2}} - 2) = -\frac{\sqrt{15} + 2\sqrt{30}}{28}$

28) $\frac{1}{2}\sqrt{\frac{1}{2}} : (\sqrt{2} + 3\sqrt{\frac{1}{2}}) = \frac{1}{10}$

29) $(1 + \sqrt{2}) : (2 - \sqrt{2}) = 2 + \frac{3}{2}\sqrt{2}$

30) $(5 - 7\sqrt{3}) : (1 + \sqrt{3}) = 6\sqrt{3} - 13$

31) $(6 - 3\sqrt{5}) : (\sqrt{5} - 1) = \frac{3}{4}\sqrt{5} - \frac{9}{4}$

32) $(\sqrt{3} + \sqrt{2}) : (\sqrt{3} - \sqrt{2}) = 5 + 2\sqrt{6}$

33) $(3\sqrt{5} - 2\sqrt{2}) : (2\sqrt{5} - \sqrt{18}) = 9 + \frac{5}{2}\sqrt{10}$

34) $(6\sqrt{7} - 3\sqrt{3}) : (\sqrt{5} - 2) = 6\sqrt{35} + 12\sqrt{7} -$
$2\sqrt{15} - 6\sqrt{3}$

35) $1 : (\sqrt{2} + \sqrt{3} - \sqrt{5}) = \frac{\sqrt{30}}{12} + \frac{\sqrt{2}}{4} + \frac{\sqrt{3}}{6}$

36) $7 : (\sqrt{10} - \sqrt{2} - \sqrt{3}) = 35\sqrt{10} + 77\sqrt{2} + 63\sqrt{3}$
$+ 14\sqrt{60}$

37) $\sqrt{2} : (1 + 2\sqrt{2} - \sqrt{5}) = \frac{3}{4}\sqrt{2} + \frac{1}{2}\sqrt{5} - \frac{1}{4}\sqrt{10} - \frac{1}{2}$

38) $(2 - \sqrt{3}) : (1 + \sqrt{2} + \sqrt{3}) = 1 + \frac{5}{4}\sqrt{2} - \frac{1}{2}\sqrt{3} - \frac{3}{4}\sqrt{6}$

39) $(3 + 4\sqrt{3}) : (\sqrt{6} + \sqrt{2} - \sqrt{5}) = \sqrt{6} + \sqrt{2} + \sqrt{5}$

40) $(156 + 12\sqrt{11}) : (6 + 14\sqrt{2} - 2\sqrt{11}) = 7\sqrt{2}$
$+ \sqrt{11} - 3$

41) $(2\sqrt{6} + 3\sqrt{10}) : (3\sqrt{2} - \sqrt{3} + \sqrt{5}) = \frac{7}{10}\sqrt{30}$
$+ \frac{27}{10}\sqrt{5} - \frac{3}{2}\sqrt{3} - 3\sqrt{2}$

42) $\sqrt{a} : (b + \sqrt{c}) = \frac{b\sqrt{a} - \sqrt{ac}}{b^2 - c}$

43) $\sqrt{a} : (\sqrt{b} + \sqrt{c}) = \frac{\sqrt{ab} - \sqrt{ac}}{b - c}$

44) $(c\sqrt{a} + d\sqrt{b}) : (f\sqrt{h} + g\sqrt{l}) =$
$\frac{cf\sqrt{ah} + df\sqrt{bh} - cg\sqrt{al} - dg\sqrt{bl}}{hf^2 - lg^2}$

45) $$[(f^2-hg^2-m)\sqrt{m}-2gm\sqrt{h}] : (f+g\sqrt{h}+\sqrt{m}) = f\sqrt{m}-g\sqrt{hm}-m$$

46) $$1 : \sqrt[n]{(a+\sqrt{b})} = \sqrt[n]{\frac{a-\sqrt{b}}{a^2-b}}$$

47) $$\sqrt[n]{(\sqrt{a}+\sqrt{b})} : \sqrt[n]{(\sqrt{a}-\sqrt{b})} = \sqrt[n]{\frac{a+b+2\sqrt{ab}}{a-b}}$$

48) $$1 : (\sqrt[4]{a}+\sqrt[4]{b}) = \frac{\sqrt[4]{a^3}-\sqrt[4]{a^2b}+\sqrt[4]{ab^2}-\sqrt[4]{b^3}}{a-b}\,*$$

49) $$(\sqrt[4]{a}+\sqrt[4]{b}):(\sqrt[4]{a}-\sqrt[4]{b}) = \frac{a+b+2\sqrt{ab}+2\sqrt[4]{a^3b}+2\sqrt[4]{ab^3}}{a-b}$$

50) $$\sqrt{(ab+\sqrt{af})} : \sqrt{a} = \sqrt{\left(b+\sqrt{\frac{f}{a}}\right)}$$

e. SQUARE ROOT OF A BINOMIAL OF THE FORM $A \pm \sqrt{B}$.

FORMULA.

$$\sqrt{(A\pm\sqrt{B})} = \sqrt{\frac{A+\sqrt{(A^2-B)}}{2}} \pm \sqrt{\frac{A-\sqrt{(A^2-B)}}{2}}$$

EXAMPLES.

1) $\sqrt{(7+4\sqrt{3})} = 2+\sqrt{3}$

2) $\sqrt{(43-15\sqrt{8})} = 5-3\sqrt{2}$

3) $\sqrt{(5-\sqrt{24})} = \sqrt{3}-\sqrt{2}$

4) $\sqrt{(3-2\sqrt{2})} = \sqrt{2}-1$

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* Multiply Divisor and Dividend by $\sqrt[4]{a}-\sqrt[4]{b}$; and proceed as usual.

5) $\sqrt{(28+5\sqrt{12})}=5+\sqrt{3}$

6) $\sqrt{(87-12\sqrt{42})}=3\sqrt{7}-2\sqrt{6}$

7) $\sqrt{(\frac{3}{2}+\sqrt{2})}=1+\frac{1}{2}\sqrt{2}$

8) $\sqrt{(2+\sqrt{3})}=\frac{1}{2}\sqrt{6}+\frac{1}{2}\sqrt{2}$

9) $\sqrt{(\sqrt{27}+2\sqrt{6})}=\sqrt[4]{12}+\sqrt[4]{3}$.*

10) $\sqrt{(\sqrt{32}-\sqrt{24})}=\sqrt[4]{18}-\sqrt[4]{2}$

11) $\sqrt{(3\sqrt{5}+\sqrt{40})}=\sqrt[4]{20}+\sqrt[4]{5}$

12) $\sqrt{(3\sqrt{6}+2\sqrt{12})}=\sqrt[4]{24}+\sqrt[4]{6}$

13) $\sqrt{(\sqrt{18}-4)}=\sqrt[4]{8}-\sqrt[4]{2}$

14) $\sqrt{(a^2+b+2a\sqrt{b})}=a+\sqrt{b}$

15) $\sqrt{(ac^2+bd^2+2cd\sqrt{ab})}=c\sqrt{a}+d\sqrt{b}$

16) $\sqrt{[2a+2\sqrt{(a^2-b^2)}]}=\sqrt{(a+b)}+\sqrt{(a-b)}$

17) $\sqrt{[x-2\sqrt{(x-1)}]}=\sqrt{(x-1)}-1$

18) $\sqrt{\left[\frac{a^2}{4}+\frac{c}{2}\sqrt{(a^2-c^2)}\right]}=\frac{c+\sqrt{(a^2-c^2)}}{2}$

19) $\sqrt{(x+xy-2x\sqrt{y})}=(\sqrt{y}-1)\sqrt{x}$

20) $\sqrt{[ap-2a\sqrt{(ap-a^2)}]}=\sqrt{(ap-a^2)}-a$

21) $\sqrt{\left[\frac{3a}{b}+\sqrt{\left(\frac{12a^3c^2}{bd^2}-\frac{4a^4c^4}{d^4}\right)}\right]}+\frac{ac}{d}+\sqrt{\left(\frac{3a}{b}-\frac{a^2c^2}{d^2}\right)}$

22) $\sqrt{[b^2-ab+\frac{a^2}{4}+\sqrt{(4ab^3-8a^2b^2+a^3b)}]}=$

$$\sqrt{ab}+\sqrt{\left(b^2-2ab+\frac{a^2}{4}\right)}$$

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* Make $A=\sqrt{27}$.

V. THE EXPRESSION OF ROOTS BY FRACTIONAL EXPONENTS, AND THEIR APPLICATION TO THE RADICAL CALCULUS.

A Power with a fractional exponent may be considered as an interpolated term of a series of powers, having integer exponents; yet the usual way in which a fractional exponent denotes the raising of a root to a power seems to be the easiest of comprehension for beginners. The whole of this theory, as well as that of logarithms, founded upon it, may be exposed with the accuracy of Euclid, and merely by signs.

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1) EXPRESSIONS.

1) $\sqrt[m]{a^n} = a^{\frac{n}{m}}$

2) $\dfrac{1}{\sqrt[m]{a^n}} = a^{-\frac{n}{m}}$

3) $\sqrt[m]{a^nb^pc^q} = a^{\frac{n}{m}}b^{\frac{p}{m}}c^{\frac{q}{m}} = (a^nb^pc^q)^{\frac{t}{m}}$

4) $\sqrt[m]{\dfrac{a^nb^p}{c^rd^se^t}} = a^{\frac{n}{m}}b^{\frac{p}{m}}c^{-\frac{r}{m}}d^{-\frac{s}{m}}e^{-\frac{t}{m}}$

5) $c\sqrt[4]{a^3} + \dfrac{d}{\sqrt[7]{a^2}} = ca^{\frac{3}{4}} + da^{-\frac{2}{7}}$

6) $\sqrt[5]{a^2bc} = a^{\frac{2}{5}}b^{\frac{1}{5}}c^{\frac{1}{5}} = (a^2bc)^{\frac{1}{5}}$

7) $\sqrt[6]{\dfrac{a^5b^7}{c^{12}}} + \sqrt[8]{\dfrac{a^6b^4}{d^{20}}} = a^{\frac{5}{6}}b^{\frac{7}{6}}c^{-2} + a^{\frac{3}{4}}b^{\frac{1}{2}}d^{-\frac{5}{2}}$

9) For Subt. $$\begin{cases} 16\sqrt[4]{6ab} - \sqrt[5]{9c^3} + 3\sqrt[m]{7a} - \sqrt{10} \\ -8\sqrt[5]{9c^3} - 5\sqrt[m]{7a} + 3\sqrt[4]{6ab} + 2\sqrt[4]{10} \end{cases}$$

$$\overline{13\sqrt[4]{6ab} + 7\sqrt[5]{9c^3} + 8\sqrt[m]{7a} - \sqrt{10} - 2\sqrt[4]{10}}$$

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b. ABBREVIATIONS AND TRANSFORMATIONS.

1) $\sqrt{24} + \sqrt{54} - \sqrt{6} = 4\sqrt{6}$

2) $2\sqrt{8} - 7\sqrt{18} + 5\sqrt{72} - \sqrt{50} = 8\sqrt{2}$

3) $\sqrt{12} + 2\sqrt{27} + 3\sqrt{75} - 9\sqrt{48} = -13\sqrt{3}$

4) $8\sqrt{\frac{3}{4}} - \frac{1}{2}\sqrt{12} + 4\sqrt{27} - 2\sqrt{\frac{3}{16}} = \frac{29}{2}\sqrt{3} = \frac{29\sqrt{3}}{2}$

5) $2\sqrt{\frac{5}{3}} + \sqrt{60} - \sqrt{15} + \sqrt{\frac{3}{5}} = \frac{28}{15}\sqrt{15}$

6) $7\sqrt[3]{54} + 3\sqrt[3]{16} + \sqrt[3]{2} - 5\sqrt[3]{128} = 8\sqrt[3]{2}$

7) $\sqrt[3]{81} - 2\sqrt[3]{24} + \sqrt{28} + 2\sqrt{63} = 8\sqrt{7} - \sqrt[3]{3}$

8) $\sqrt[4]{32} + 2\sqrt[3]{40} = 2\sqrt[4]{2} + 4\sqrt[3]{5}$

9) $3\sqrt{5} - 2\sqrt{2} + 3\sqrt{6} = \sqrt{45} - \sqrt{8} + \sqrt{54}$

10) $5\sqrt[3]{7} + 3\sqrt{2} + 2\sqrt[4]{3} = \sqrt[3]{875} + \sqrt{18} + \sqrt[4]{48}$

11) $4\sqrt[5]{\frac{1}{2}} + 3\sqrt[3]{2} - 5\sqrt{\frac{1}{3}} = \sqrt[5]{512} + \sqrt[3]{54} - \sqrt{\frac{25}{3}}$

12) $\sqrt{45c^3} - \sqrt{80c^3} + \sqrt{5a^2c} = (a - c)\sqrt{5c}$

13) $\sqrt{18a^5b^3} + \sqrt{50a^3b^3} = (3a^2b + 5ab)\sqrt{2ab}$

14) $\sqrt[3]{16a^3b} + \sqrt{4a^2b} - \sqrt{a^2b} - \sqrt[3]{54a^3b} = a\sqrt{b} - a\sqrt[3]{2b}$

15) $\sqrt[4]{2^{14}a^{13}b^5c} - \sqrt[4]{4 \cdot 5^4a^5b^9c^5} + \sqrt[4]{4 \cdot 6^4ab^5c} =$
$(8a^3b - 5ab^2c + 6b)\sqrt[4]{4abc}$

16) $\sqrt{\frac{a^4c}{b^3}} + \sqrt{\frac{a^2c^3}{bd^2}} - \sqrt{\frac{a^2cd^2}{be^2}} = \left(\frac{a^2}{b} + \frac{ac}{d} - \frac{ad}{e}\right)\sqrt{\frac{c}{b}}$

17) $\sqrt[3]{\frac{27a^5x}{2b}} - \sqrt[3]{\frac{a^2x}{2b}} = (3a - 1)\sqrt[3]{\frac{a^2x}{2b}}$

18) $3b^2\sqrt{a^3c} + \frac{2}{c}\sqrt{a^5c^3} - c^4\sqrt{\frac{ac}{b^2}} = \left(3ab^2 + 2a^2 - \frac{c^4}{b}\right)\sqrt{ac}$

19) $5a\sqrt[3]{\frac{a^2}{b}} + b\sqrt[3]{\frac{b^2c^3}{a}} = \left(\frac{5a}{b} + \frac{bc}{a}\right)\sqrt[3]{a^2b^2}$

20) $\sqrt[3]{54a^{m+6}b^3} - \sqrt[3]{16a^{m-3}b^6} + \sqrt[3]{2a^{4m+9}} + \sqrt[3]{2c^3a^m}$
$= \left(3a^2b - \frac{2b^2}{a} + a^{m+3} + c\right)\sqrt[3]{2a^m}$

21) $\sqrt[m]{2^ma^{mp+3}b^{mn+5}} + \sqrt[m]{3^ma^{2m-mn+3}b^{m+5}} - \sqrt[m]{a^3b^5c^{2m}}$
$= (2a^pb^n + 3a^{2-n}b - c^2)\sqrt[m]{a^3b^5}$

22) $\sqrt[6]{\frac{3 \cdot 2^3c^3f^4}{d^4g}} + \sqrt[6]{\frac{2^3g^{11}}{3^5c^3d^4f^2}} = \left(\frac{f}{d} + \frac{g^2}{3cd}\right)\sqrt[6]{\frac{3 \cdot 2^3c^3d^2}{f^2g}}$

23) $\sqrt[2n]{\frac{a^{6n+4}b^{2n-3}c^{2mn}}{d^{9n+5}f^2g^{p+2n-1}}} = \frac{a^3bc^m}{d^4g}\sqrt[2n]{\frac{a^4}{d^{n+5}f^2g^{p-1}b^3}}$

24) $\sqrt{(a^2c + a^2d)} = a\sqrt{(c + d)}$

25) $\sqrt[3]{(a^{6m}b - a^{7m}f^2)} = a^{2m}\sqrt[3]{(b - a^mf^2)}$

26) $\sqrt[7]{\left(\frac{a^{10}}{b^6c^6} + \frac{a^8}{b^5c^5}\right)} = \frac{a}{bc}\sqrt[7]{(a^3bc + ab^2c^2)}$

27) $\sqrt{\left(\frac{a^3b^2}{cd^2} - \frac{2a^2b^3}{c^2d}\right)} = \frac{ab}{cd}\sqrt{(ac - 2bd)}$

28) $x\sqrt[3]{\left(\frac{8a^4}{27b^3} + \frac{16a^3}{27b^2}\right)} = \frac{2ax}{3b}\sqrt[3]{(a + 2b)}$

29) $\sqrt{(3a^2c + 6abc + 3b^2c)} = (a + b)\sqrt{3c}$

30) $\sqrt{(4a^5b^2 - 20a^3b^3 + 25ab^4)} = (2a^2 - 5b)\sqrt{ab^2}$

31) $\sqrt{(2ax^2 - 4ax + 2a)} = (x - 1)\sqrt{2a}$

32) $\sqrt{\dfrac{a^3b - 4a^2b^2 + 4ab^3}{c^2d^2}} = \dfrac{a - 2b}{cd}\sqrt{ab}$

33) $\sqrt{\dfrac{a^2x - 2ax^2 + x^3}{a^2 + 2ax + x^2}} = \dfrac{a - x}{a + x}\sqrt{x}$

34) $\sqrt{\dfrac{ac}{a^2bd - 2ab^2d + b^3d}} = \dfrac{1}{a - b}\sqrt{\dfrac{ac}{bd}}$

35) $\sqrt{\dfrac{x^3 + 2x^2 + x}{a^3 + a^2b}} = \dfrac{x + 1}{a}\sqrt{\dfrac{x}{a + b}}$

36) $\sqrt{\dfrac{a^3 - a^2x - ax^2 + x^3}{b^5c^3d}} = \dfrac{a - x}{b^2c}\sqrt{\dfrac{a + x}{bcd}}$

37) $\dfrac{a - b}{a + b}\sqrt{\dfrac{ac}{a^2 - 2ab + b^2}} = \dfrac{\sqrt{ac}}{a + b}$

38) $\dfrac{a + b}{a - b}\sqrt{\dfrac{a - b}{a + b}} = \sqrt{\dfrac{a + b}{a - b}}$

39) $(x + 1)\sqrt{\dfrac{f^2g}{x^2 - 1}} = \sqrt{\dfrac{(x + 1)^2f^2g}{(x + 1)(x - 1)}} = \sqrt{\dfrac{(x + 1)f^2g}{x - 1}}$

40) $\sqrt[m]{\sqrt[n]{\sqrt[p]{\sqrt[q]{a}}}} = \sqrt[mnpq]{a}$

41) $\sqrt[3]{(2\sqrt{5})} = \sqrt[6]{20}$

42) $\sqrt[m]{(a\sqrt[n]{b})} = \sqrt[mn]{a^nb}$

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c. MULTIPLICATION.

1) $\sqrt[m]{a} \times \sqrt[m]{b} \times \sqrt[m]{c} = \sqrt[m]{abc}$

2) $a\sqrt[n]{x} \times b\sqrt[n]{y} \times c\sqrt[n]{z} = abc\sqrt[n]{xyz}$

3) $\sqrt[3]{4} \times 7\sqrt[3]{6} \times \frac{1}{2}\sqrt[3]{5} = \frac{7}{2}\sqrt[3]{120}$

4) $4 \times 2\sqrt[6]{3} \times \sqrt[6]{72} = 8\sqrt{6}$

7) $h : \dfrac{a^{\frac{1}{3}}b^{\frac{1}{4}}}{cd^{\frac{1}{2}}} = \dfrac{ch\sqrt{d}}{\sqrt[12]{a^4b^3}}$

8) $\dfrac{a^{-\frac{9}{2}}b^{\frac{2}{3}}}{c^{\frac{1}{6}}d^3} : \dfrac{a^{-\frac{29}{4}}d^{\frac{11}{3}}}{b^{\frac{8}{5}}c} = \dfrac{a^{\frac{11}{4}}b^{\frac{34}{15}}c}{c^{\frac{1}{6}}d^{\frac{20}{3}}} = \dfrac{a^3b^2c}{d^7}\sqrt[60]{\dfrac{b^{16}d^{20}}{a^{15}c^{10}}} =$

$\dfrac{a^3b^2c}{d^7}\sqrt[12]{\dfrac{d^4}{a^3c^2}} \cdot \sqrt[15]{b^4}$

9) $(a^3 - 2\sqrt[4]{a^2b^3} - a^2\sqrt[6]{a^3b^2} + 2b\sqrt[12]{b}) : (\sqrt{a} - \sqrt[3]{b}) =$

$(a^3 - 2a^{\frac{1}{2}}b^{\frac{3}{4}} - a^{\frac{5}{2}}b^{\frac{1}{3}} + 2b^{\frac{13}{12}}) : (a^{\frac{1}{2}} - b^{\frac{1}{3}}) =$

$a^{\frac{5}{2}} - 2b^{\frac{3}{4}} = a^2\sqrt{a} - 2\sqrt[4]{b^3}$

10) $\left(\sqrt[12]{a^{10}b^9} - c\sqrt[10]{a^7} \cdot \sqrt[6]{b^5} - \frac{3}{2}a\sqrt[4]{b^3} + \dfrac{3abc}{2}\sqrt[30]{\dfrac{1}{a^4b^5}}\right)$

$: (\sqrt{ab} - \frac{3}{2}\sqrt[6]{a^4b^3}) = (a^{\frac{5}{6}}b^{\frac{3}{4}} - ca^{\frac{7}{10}}b^{\frac{5}{6}} - \frac{3}{2}ab^{\frac{3}{4}}$

$+ \frac{3}{2}ca^{\frac{13}{15}}b^{\frac{5}{6}}) : (a^{\frac{1}{2}}b^{\frac{1}{2}} - \frac{3}{2}a^{\frac{2}{3}}b^{\frac{1}{2}}) = a^{\frac{1}{3}}b^{\frac{1}{4}} - ca^{\frac{1}{5}}b^{\frac{1}{3}}$

$= \sqrt[12]{a^4b^3} - c\sqrt[15]{a^3b^5}$

11) $(5a^2 - 41ab + 42b^2)\sqrt[12]{a} : \left(\sqrt[3]{a} - \dfrac{7b}{\sqrt[3]{a^2}}\right)$

$= (5a^{\frac{25}{12}} - 41a^{\frac{13}{12}}b + 42a^{\frac{1}{12}}b^2) : (a^{\frac{1}{3}} - 7ba^{-\frac{2}{3}})$

$= 5a^{\frac{7}{4}} - 6a^{\frac{3}{4}}b = (5a - 6b)\sqrt[4]{a^3}$

12) $(\sqrt[4]{a^3} - \sqrt[4]{b^3}) : (\sqrt[4]{a} - \sqrt[4]{b}) = (a^{\frac{3}{4}} - b^{\frac{3}{4}}) : (a^{\frac{1}{4}} - b^{\frac{1}{4}})$

$= a^{\frac{1}{2}} + a^{\frac{1}{4}}b^{\frac{1}{4}} + b^{\frac{1}{2}} = \sqrt{a} + \sqrt{b} + \sqrt[4]{ab}$

c. POWERS OF POWERS.

$$1)\quad \left(a^{\frac{m}{n}}\right)^{\frac{p}{q}} = \sqrt[q]{\left(\sqrt[n]{a^m}\right)^p} = a^{\frac{mp}{nq}} = \sqrt[nq]{a^{mp}}$$

$$2)\quad \left(a^{-\frac{m}{n}}\right)^{\frac{p}{q}} = \sqrt[q]{\left(\frac{1}{\sqrt[n]{a^m}}\right)^p} = a^{-\frac{mp}{nq}} = \frac{1}{\sqrt[nq]{a^{mp}}}$$

$$3)\quad \left(a^{\frac{m}{n}}\right)^{-\frac{p}{q}} = \frac{1}{\sqrt[q]{\left(\sqrt[n]{a^m}\right)^p}} = a^{-\frac{mp}{nq}} = \frac{1}{\sqrt[nq]{a^{mp}}}$$

$$4)\quad \left(a^{-\frac{m}{n}}\right)^{-\frac{p}{q}} = \frac{1}{\sqrt[q]{\left(\frac{1}{\sqrt[n]{a^m}}\right)^p}} = a^{\frac{mp}{nq}} = \sqrt[nq]{a^{mp}}$$

$$5)\quad \left(a^{\frac{3}{4}}b^{\frac{2}{3}}\right)^{\frac{1}{3}} = a^{\frac{1}{4}}b^{\frac{2}{9}} = \sqrt[36]{a^9b^8}$$

$$6)\quad \left(a^2b^{-\frac{1}{2}}c^{\frac{2}{5}}\right)^{-\frac{1}{4}} = a^{-\frac{1}{2}}b^{\frac{1}{8}}c^{-\frac{1}{10}} = \sqrt[40]{\frac{b^5}{a^{20}c^4}}$$

$$7)\quad \left[\left(a^{\frac{1}{2}}\right)^{\frac{3}{5}}\right]^{-\frac{1}{6}} = a^{-\frac{1}{20}} = \sqrt[20]{\frac{1}{a}}$$

$$8)\quad \sqrt[6]{\left(a^3b\sqrt[5]{a^3bc}\right)^5} = \left(a^{\frac{18}{5}}b^{\frac{6}{5}}c^{\frac{1}{5}}\right)^{\frac{5}{6}} = a^3bc^{\frac{1}{6}} = a^3b\sqrt[6]{c}$$

$$9)\quad \left[\frac{c^2d}{(a+b)^{\frac{3}{2}}}\right]^{-\frac{1}{3}} = \frac{c^{-\frac{2}{3}}d^{-\frac{1}{3}}}{(a+b)^{-\frac{1}{2}}} = \frac{\sqrt{(u+b)}}{\sqrt[3]{c^2d}} = \sqrt[6]{\frac{(a+b)^3}{c^4d^2}}$$

$$10)\quad \sqrt[4]{\left(\frac{a\sqrt{b}}{\sqrt[3]{ab}}\right)^3} = \left(a^{\frac{2}{3}}b^{\frac{1}{6}}\right)^{\frac{3}{4}} = a^{\frac{1}{2}}b^{\frac{1}{8}} = \sqrt[8]{a^4b}$$

$$11)\quad \sqrt[4]{\frac{\sqrt{(c-d)}\cdot\sqrt[3]{(a+x^2)^4}}{c^6d^{5m}}} = \left[\frac{(c-d)^{\frac{1}{2}}(a+x^2)^{\frac{4}{3}}}{c^6d^{5m}}\right]^{\frac{1}{4}} =$$

$$\frac{(c-d)^{\frac{1}{8}}(a+x^2)^{\frac{1}{3}}}{c^{\frac{3}{2}}d^{\frac{5m}{4}}} = \frac{1}{cd^m}\sqrt[24]{\frac{(c-d)^3\,(a+x^2)^8}{c^{12}d^{6m}}}$$

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42) $(\sqrt[3]{5} - 2\sqrt[3]{6}) \times (3\sqrt[3]{4} - \sqrt[3]{36}) = 12 + 3\sqrt[3]{20}$
$- 6\sqrt[3]{24} - \sqrt[3]{180}$

43) $(5\sqrt[3]{4} - 2\sqrt[3]{16}) \times (2\sqrt[3]{2} - 3\sqrt[3]{4}) = 44 - 4\sqrt[3]{32}$
$- 15\sqrt[3]{16}$

44) $(2\sqrt{3} + \sqrt[3]{2}) \times (2 + \sqrt[3]{9}) = 4\sqrt{3} + 2\sqrt[3]{2} + \sqrt[3]{18}$
$+ 6\sqrt[6]{3}$

45) $(5 + \sqrt[3]{4} + 2\sqrt[4]{5}) \times (\sqrt{6} + \sqrt{5}) = 5\sqrt{6} + 5\sqrt{5}$
$+ 2\sqrt[4]{125} + 2\sqrt[4]{180} + 2\sqrt[6]{54} + \sqrt[6]{2000}$

46) $(a + \sqrt{b}) \times (a - \sqrt{b}) = a^2 - b$

47) $(\sqrt{a} + \sqrt{b}) \times (\sqrt{a} - \sqrt{b}) = a - b$

48) $(c\sqrt{a} + d\sqrt{b}) \times (c\sqrt{a} - d\sqrt{b} = ac^2 - bd^2$

49) $(a + \sqrt{x}) \times (b + \sqrt{y}) = ab + a\sqrt{y} + b\sqrt{x} + \sqrt{xy}$

50) $\left(\sqrt{\frac{ad^2}{c^3}} + \sqrt{\frac{a^2}{b}}\right) \times (\sqrt{ac} + \sqrt{b^3}) = \frac{ad}{c} + ab$
$+ \left(a + \frac{b^2 d}{c^2}\right)\sqrt{\frac{ac}{b}}$

51) $\left(\sqrt{\frac{ac^2}{a+b}} + \sqrt{\frac{1}{b}}\right) \times \left(\frac{c}{d}\sqrt{(a+b)\,a} - \sqrt{\frac{b^5}{c^2}}\right) =$
$\frac{ac^2}{d} - \frac{b^2}{c} + \left(\frac{c}{bd} - \frac{b^2}{a+b}\right)\sqrt{(a+b)ab}$

52) $(\sqrt{a} + c\sqrt[3]{b}) \times (\sqrt{a} - c\sqrt[3]{b}) = a - c^2\sqrt[3]{b^2}$

53) $(2\sqrt{a} + 3c\sqrt[3]{b}) \times (\sqrt{a} + 4\sqrt[3]{b}) = 2a + 12c\sqrt[3]{b^2}$
$+ (3c + 8)\sqrt[6]{a^3 b^2}$

54) $(c\sqrt[4]{a} + d\sqrt[4]{b}) \times (f\sqrt[4]{a} + g\sqrt[4]{b}) = cf\sqrt{a} + dg\sqrt{b}$
$+ (df + cg)\sqrt[4]{ab}$

2) MULTIPLICATION.

1) $a \times \sqrt{-a} = a\sqrt{a}\cdot\sqrt{-1}$

2) $c\sqrt{-a} \times d\sqrt{-b} = c\sqrt{a}\cdot\sqrt{-1} \times d\sqrt{b}\cdot\sqrt{-1} = -cd\sqrt{ab}$

3) $(c\sqrt{-a}+d\sqrt{-b}+f)\times\sqrt{-a} = -ac-d\sqrt{ab}+f\sqrt{a}\cdot\sqrt{-1}$

4) $(2-\sqrt{-3})\times(10-\sqrt{-8}) = 20-\sqrt{24}-(10\sqrt{3}+4\sqrt{2})\sqrt{-1}$

5) $(7-\sqrt{-5})\times(10-3\sqrt{-6}) = 70-3\sqrt{30}-(10\sqrt{5}+21\sqrt{6})\sqrt{-1}$

6) $(3-\sqrt{-5})\times(4-2\sqrt{-5}) = 2-10\sqrt{5}\cdot\sqrt{-1}$

7) $(2-5\sqrt{-3})\times(7-4\sqrt{-3}) = -46-43\sqrt{3}\cdot\sqrt{-1}$

8) $(9+6\sqrt{-1})\times(3+7\sqrt{-1}) = -15+81\sqrt{-1}$

9) $(7-\sqrt{-\tfrac{1}{2}})\times(1-\sqrt{-\tfrac{1}{2}}) = \tfrac{13}{2}-4\sqrt{2}\cdot\sqrt{-1}$

10) $(1-\sqrt{-1})^2 = -2\sqrt{-1}$

11) $(\sqrt{2}-3\sqrt{-5})\times(\sqrt{7}-\sqrt{-3}) = \sqrt{14}-3\sqrt{15}-(3\sqrt{35}+\sqrt{6})\sqrt{-1}$

12) $(2\sqrt{3}-\sqrt{-5})\times(4\sqrt{3}-2\sqrt{-5}) = 14-8\sqrt{-15}$

13) $(2\sqrt{-3}-5\sqrt{-4}-7\sqrt{-2})\times(\sqrt{-7}-2\sqrt{-1}) = -2\sqrt{21}+5\sqrt{28}+7\sqrt{14}+4\sqrt{3}-20-14\sqrt{2}$

14) $(\sqrt{a}\cdot\sqrt{-1}+\sqrt{b}\cdot\sqrt{-1})^2 = -(a+b+2\sqrt{ab})$

15) $(a+\sqrt{b}\cdot\sqrt{-1})\times(a-\sqrt{b}\cdot\sqrt{-1}) = a^2+b$

16) $(a\pm\sqrt{b}\cdot\sqrt{-1})^2 = a^2-b\pm2a\sqrt{b}\cdot\sqrt{-1}$

17) $(a\pm\sqrt{b}\cdot\sqrt{-1})^3 = a^3-3ab\pm(3a^2\sqrt{b}-b\sqrt{b})\sqrt{-1}$

18) $(a\sqrt{-1})^{4n} = a^{4n}$

19) $(a\sqrt{-1})^{4n+1} = a^{4n+1}\sqrt{-1}$

20) $(a\sqrt{-1})^{4n+2} = -a^{4n+2}$

21) $(a\sqrt{-1})^{4n+3} = -a^{4n+3}\sqrt{-1}$

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3) DIVISION.

1) $b\sqrt{-1} : c\sqrt{-1} = \frac{b}{c}$

2) $1 : \sqrt{-1} = -\sqrt{-1}$

3) $a : b\sqrt{-1} = -\frac{a}{b}\sqrt{-1}$

4) $a : \sqrt{a} \cdot \sqrt{-1} = -\sqrt{a} \cdot \sqrt{-1}$

5) $(\sqrt{-12} + \sqrt{-6} + \sqrt{-9}) : \sqrt{-3}$
$= 2 + \sqrt{2} + \sqrt{3}$

6) $(2\sqrt{8} - \sqrt{-10}) : -\sqrt{-2} = \sqrt{5} - 4\sqrt{-1}$

7) $(3\sqrt{-4} - 2\sqrt{-12} + \sqrt{6} - 9) : -3\sqrt{-2}$
$= -\sqrt{2} + \frac{2}{3}\sqrt{6} + \left(\frac{1}{3}\sqrt{3} - \frac{3}{\sqrt{2}}\right)\sqrt{-1}$

8) $6 : (1 + \sqrt{-2}) = 2 - 2\sqrt{2} \cdot \sqrt{-1}$

9) $8 : (-1 + \sqrt{-3}) = -2 - 2\sqrt{3} \cdot \sqrt{-1}$

10) $1 : (3 - 2\sqrt{-3}) = \frac{3 + 2\sqrt{3} \cdot \sqrt{-1}}{21}$

11) $14 : (4\sqrt{-3} - 2\sqrt{-5}) = -(2\sqrt{3} + \sqrt{5})\sqrt{-1}$

12) $(5 - \sqrt{-2}) : (1 + \sqrt{-2}) = 1 - 2\sqrt{2} \cdot \sqrt{-1}$

13) $(4\sqrt{5} - 20) : (\frac{3}{2}\sqrt{-10} - 5\sqrt{-\frac{1}{2}}) =$
$(\sqrt{10} + \sqrt{2})\, 2\sqrt{-1}$

14) $[14-\sqrt{15}-(7\sqrt{3}+2\sqrt{5})\sqrt{-1}]:(7-\sqrt{5}\cdot\sqrt{-1})$
$=2-\sqrt{3}\cdot\sqrt{-1}$

15) $1:[2+(\sqrt{3}-\sqrt{5})\sqrt{-1}]$
$$=\frac{12+2\sqrt{15}+(3\sqrt{5}-\sqrt{3})\sqrt{-1}}{42}$$

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4) SQUARE ROOT OF A BINOMIAL OF THE FORM $A \pm B\sqrt{-1}$.

FORMULA.

$$\sqrt{(A \pm B\sqrt{-1})}$$
$$=\sqrt{\frac{\sqrt{(A^2+B^2)}+A}{2}} \pm \sqrt{\frac{\sqrt{(A^2+B^2)}-A}{2}}\cdot\sqrt{-1}$$

EXAMPLES.

1) $\sqrt{(7+6\sqrt{-2})}=\sqrt{(7+6\sqrt{2}.\sqrt{-1})}=$
$3+\sqrt{2}.\sqrt{-1}$

2) $\sqrt{(31+42\sqrt{-2})}=7+3\sqrt{2}.\sqrt{-1}$

3) $\sqrt{(16-24\sqrt{-5})}=6-2\sqrt{5}.\sqrt{-1}$

4) $\sqrt{(-3+\sqrt{-16})}=1+2\sqrt{-1}$

5) $\sqrt{(4\sqrt{-6}-2)}=2+\sqrt{6}.\sqrt{-1}$

6) $\sqrt{(-83-60\sqrt{-3})}=5-6\sqrt{3}.\sqrt{-1}$

7) $\sqrt{(2+4\sqrt{-42})}=\sqrt{14}+2\sqrt{3}.\sqrt{-1}$

8) $\sqrt{(-2-2\sqrt{-15})}=\sqrt{3}-\sqrt{5}.\sqrt{-1}$

9) $\sqrt{\left(\frac{a^2c}{b^2}-cd+\frac{ac\sqrt{4d}}{b}\sqrt{-1}\right)}=\frac{a}{b}\sqrt{c}+\sqrt{cd}.\sqrt{-1}$

10) $\sqrt{\left(\frac{25a^2d}{c^2} - \frac{4a^2b}{d} - \frac{20a^2\sqrt{b}}{c}\sqrt{-1}\right)} = \frac{5a\sqrt{d}}{c} - 2a\sqrt{\frac{b}{d}}\cdot\sqrt{-1}$

11) $\sqrt{[a^4f^4 - a^3b^2 - a^2b^3 - 2a^3bf^2\sqrt{(a+b)}\cdot\sqrt{-1}]}$
$= a^2f^2 - ab\sqrt{(a+b)}\cdot\sqrt{-1}$

12) $\sqrt[4]{-1} = \sqrt{(0+\sqrt{-1})} = \sqrt{\tfrac{1}{2}} + \sqrt{\tfrac{1}{2}}\cdot\sqrt{-1}$

13) $\sqrt{(-\sqrt{-1})} = \sqrt{(0-\sqrt{-1})} = \sqrt{\tfrac{1}{2}} - \sqrt{\tfrac{1}{2}}\cdot\sqrt{-1}$

14) $\sqrt{(8\sqrt{-1})} = \sqrt{(0+8\sqrt{-1})} = 2 + 2\sqrt{-1}$

15) $\sqrt{\left(\frac{2c^2}{d^2}\cdot\sqrt{-1}\right)} = \frac{c}{d}(1+\sqrt{-1})$

16) $\sqrt{(2cd\sqrt{-1})} = (1+\sqrt{-1})\sqrt{cd}$

17) $\sqrt{(2+\sqrt{-3})} = \sqrt{\frac{\sqrt{7}+2}{2}} + \sqrt{\frac{\sqrt{7}-2}{2}}\cdot\sqrt{-1}$

18) $\sqrt{(5-\sqrt{-1})} = \sqrt{\frac{\sqrt{26}+5}{2}} - \sqrt{\frac{\sqrt{26}-5}{2}}\cdot\sqrt{-1}$

VII. REDUCTIONS.

1) REDUCTION OF FRACTIONS TO A COMMON DENOMINATOR.

1) $\frac{a}{b} + c = \frac{a+bc}{b}$

2) $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$

3) $\frac{a}{b} + \frac{c}{d} + \frac{e}{f} = \frac{adf+bcf+bde}{bdf}$

4) $\frac{3a}{5b} + \frac{c}{4d} + h = \frac{12ad + 5bc + 20bdh}{20bd}$

5) $\frac{a}{b} + \frac{c}{d} - \frac{e}{f} - \frac{g}{h} - k = \frac{adfh + bcfh - bdeh - bdfg - bdfhk}{bdfh}$

6) $\frac{1}{a} - \frac{1}{b} + \frac{1}{c} = \frac{bc - ac + ab}{abc}$

7) $\frac{3a}{4b} + \frac{5f}{8l} - \frac{x}{7y} = \frac{42aly + 35bfy - 8blx}{56bly}$

8) $\frac{af}{4bg} - \frac{5cd}{12bh} + \frac{2}{3} = \frac{3afh - 5cdg + 8bgh}{12bgh}$

9) $\frac{a}{4bcd} - \frac{h}{2bcg} + \frac{2cd}{5bg} = \frac{5ag - 10dh + 8c^2d^2}{20bcdg}$

10) $\frac{2a}{3bc} + \frac{5df}{8b^2c} - \frac{deg}{6b^2c^2} = \frac{16abc + 15cdf - 4deg}{24b^2c^2}$

11) $a - b - \frac{d}{ef} - \frac{c}{eg} = \frac{(a-b)efg - dg - cf}{efg}$

12) $e - f - \frac{g^3}{2ef} + \frac{f^m}{3eg} = \frac{6efg(e-f) - 3g^4 + 2f^{m+1}}{6efg}$

13) $\frac{a^2d}{3b^7c^3} - \frac{3ad}{2b^4c^2} - \frac{b^2}{cd} = \frac{2a^2d^2 - 9ab^3cd^2 - 6b^9c^2}{6b^7c^3d}$

14) $\frac{a}{b^n} + \frac{c}{b^{n-r}} + \frac{d}{b^{n-2r}} = \frac{a + cb^r + db^{2r}}{b^n}$

15) $\frac{a}{x^n} - \frac{c}{x^{n-1}} \frac{d}{x^{n-r-s}} = \frac{a - cx + dx^{r+s}}{x^n}$

16) $c + 2ab - 3ac - \frac{b^2c - 5ab^2c + a^3}{b^2 - bc} =$

$$\frac{2ab^3 - bc^2 + 3abc^2 - a^3}{b^2 - bc}$$

17) $\frac{a+b}{2}+\frac{a-b}{2}=a$

18) $\frac{a+b}{2}-\frac{a-b}{2}=b$

19) $\frac{13a-5b}{4}-\frac{7a-2b}{6}-\frac{3a}{5}=\frac{89a-55b}{60}$

20) $\frac{3a-4b}{7}-\frac{2a-b-c}{3}+\frac{15a-4c}{12}=\frac{85a-20b}{84}$

21) $\frac{3a+2b}{c}-\frac{5bd-2a-3d}{4cd}=\frac{12ad+3bd+2a+3d}{4cd}$

22) $\frac{a}{b}+\frac{a-3b}{cd}+\frac{a^2-b^2-ab}{bcd}=\frac{acd-4b^2+a^2}{bcd}$

23) $cf+\frac{a^2}{c^5f}-\frac{a-b-c^2}{bc^5f^3}=\frac{bc^6f^4+a^2bf^2-a+b+c^2}{bc^5f^3}$

24) $\frac{3a+b+x}{5a}-\frac{2a+b}{3b}+\frac{7a-2b}{9a}=\frac{47ab-b^2+9bx-30a^2}{45ab}$

25) $\frac{3a^m(a+b)^{m-2}}{c^{m+2}d^{m-3}f^4}-\frac{a^{3m}-2acd^{4-m}}{c^{m+1}df^n(a+b)^2}-\frac{1}{c^{m-2}f^{n-3}(a+b)^2}$

$=\frac{3a^mf^{n-4}(a+b)^m-ca^{3m}d^{m-4}+2ac^2-c^4f^3d^{m-3}}{c^{m+2}d^{m-3}f^n(a+b)^2}$

26) $\frac{(a+x)^{\frac{p}{q}-1}}{3b^2(c+x)^{\frac{m}{n}}}-\frac{b^{\frac{2}{3}}x^2(c+x)^{-\frac{m}{n}}}{(a+x)^{1-\frac{p}{2q}}}=\frac{(a+x)^{\frac{}{2q}}-3b^{\frac{8}{3}}x^2}{3b^2(c+x)^{\frac{m}{n}}\cdot(a+x)^{1-\frac{p}{2q}}}$

27) $\frac{a}{a+z}+\frac{z}{a-z}=\frac{a^2+z^2}{a^2-z^2}$

28) $\frac{f+g}{2f-2g}-\frac{5f-2g}{2f-9g}=\frac{9fg-13f^2-13g^2}{6f^2-13fg+18g^2}$

29) $\frac{a}{b+x}-\frac{c}{x}+\frac{3c}{4x}+2b=\frac{8bx^2+(8b^2+4a-c)x-bc}{4bx+4x^2}$

30) $\frac{3a+2x}{a+x}-\frac{5a-x}{a-x}+\frac{a}{2x}=\frac{a^3-4a^2x-11ax^2-2x^3}{2x(a^2-x^2)}$

31) $\frac{az}{a^2-z^2}-\frac{a-z}{a+z}=\frac{3az-a^2-z^2}{a^2-z^2}$

32) $\frac{ac}{a^2-4y^2}+\frac{bd}{ac+2cy}=\frac{ac^2+abd-2bdy}{c(a^2-4y^2)}$

33) $\frac{a^3}{(a+b)^3}-\frac{ab}{(a+b)^2}+\frac{b}{a+b}=\frac{a^3+ab^2+b^3}{(a+b)^3}$

34) $\frac{a^m}{(a+b)^n}+\frac{a^{m-2}b^r}{(a+b)^{n-1}}-\frac{a^{m-3}b^r}{(a+b)^{n-2}}$

$=\frac{a^m-a^{m-2}b^{r+1}-a^{m-3}b^{r+2}}{(a+b)^n}$

35) $\frac{2ax+x^2}{(a-x)^2}-\frac{a^2+5ax}{(a+x)^2}-\frac{x}{a-x}=\frac{2x^4+13a^2x^2-2a^3x-a^4}{(a^2-x^2)^2}$

36) $\frac{a-(n+1)a^{n+1}}{1-a}+\frac{a^2(1-a^n)}{(1-a)^2}=\frac{a-(n+1)a^{n+1}+na^{n+2}}{(1-a)^2}$

37) $\frac{1}{1-z^2}-\frac{1}{m+1+(m-1)z^2}=\frac{m(1+z^2)}{m(1-z^4)+(1-z^2)^2}$

38) $\frac{3}{4(1-x)^2}+\frac{3}{8(1-x)}+\frac{1}{8(1+x)}-\frac{1-x}{4(1+x^2)}$

$=\frac{1+x+x^2}{1-x-x^4+x^5}$

39) $\frac{1+2x^3}{(3-x)(1+x)}+\frac{7}{(2+x)(1-3x)}+\frac{x}{(1+x)(2+x)}$

$=\frac{23+16x-30x^2-3x^3}{(3-x)(1+x)(2+x)(1-3x)}$

40) $\frac{3h}{(h-2x)^2}+\frac{2h+x}{(h+x)(h-2x)}-\frac{5}{h+x}=$

$$\frac{20hx-22x^2}{(h+x)(h-2x)^2}$$

---

2) REDUCTION OF FRACTIONS BY SUPPRESSING THE EQUAL FACTORS IN THE NUMERATOR AND DENOMINATOR.*

1) $\frac{ax+x^2}{3bx-cx}=\frac{a+x}{3b-c}$

2) $\frac{ac^3-bc^5-c^7}{3bc^2+c^4}=\frac{ac-bc^3-c^5}{3b+c^2}$

3) $\frac{21a^3b^2c-9ab^3c^2}{15a^2b^2c+3a^5b^4c^2-12ab^2c}=\frac{7a^2-3bc}{5a+a^4b^2c-4}$

4) $\frac{2a^{n+r}b^{m-1}c-4a^rb^{2m-1}c^2d+2a^{r+1}b^mc+6a^{r-1}b^{m-1}c^n}{8a^{+5}b^{m+2}c^2-2a^{r+3}b^mc+10a^rb^3c^4}$

$$=\frac{a^n-2b^mcd+ab+3a^{-1}c^{n-1}}{4a^5b^3c-a^3b+5b^{4-m}c^3}$$

---

* This reduction presupposes the knowledge of finding the greatest common divisor of two numbers.—(See the examples, pages 8, 9, 10.) A similar method applies also to Algebraic expressions; but it leads frequently to very diffuse and intricate expressions. The solution of equations of higher degrees teaches how to find the factors of Algebraic expressions; and may sometimes be used with advantage. Practice, and due attention paid to the nature of Algebraic expressions, are the best means of acquiring skill in decomposing them into factors.

5) $\dfrac{14a^2 - 7ab}{10ac - 5bc} = \dfrac{7a}{5c}$

6) $\dfrac{12a^3x^4 + 2a^2x^5}{18ab^2x + 3b^2x^2} = \dfrac{2a^2x^3}{3b^2}$

7) $\dfrac{6ac + 9bc - 5c^2}{12adf + 18bdf - 10cdf} = \dfrac{c}{2df}$

8) $\dfrac{45a^3b^4c + 27a^8b^7cd - 9a^4b^2d^3}{30a^2b^2c^3d^4 + 18a^7b^5c^3d^5 - 6a^3c^2d^7} = \dfrac{3ab^2}{4c^2d^4}$

9) $\dfrac{30a^{3n-1}b^rc^{r+2} - 6a^{2n-4}b^3c^rd^{r-1}}{20a^nb^{r-1}c^2d^2 - 4a^{-3}b^2d^{r+1}} = \dfrac{3a^{2n-1}bc^r}{2d^2}$

10) $\dfrac{5a^2 + 5ax}{a^2 - x^2} = \dfrac{5a}{a-x}$

11) $\dfrac{a^3 - x^3}{(a-x)^2} = \dfrac{a^2 + ax + x^2}{a - x}$

12) $\dfrac{n^2 - 2n + 1}{n^2 - 1} = \dfrac{n - 1}{n + 1}$

13) $\dfrac{a^3 + (1 + a)ay + y^2}{a^4 - y^2} = \dfrac{a + y}{a^2 - y}$

14) $\dfrac{ac + bd + ad + bc}{af + 2bx + 2ax + bf} = \dfrac{c + d}{f + 2x}$

15) $\dfrac{6ac + 10bc + 9ad + 15bd}{6c^2 + 9cd - 2c - 3d} = \dfrac{3a + 5b}{3c - 1}$

16) $\dfrac{n^3 - 2n^2}{n^2 - 4n + 4} = \dfrac{n^2}{n - 2}$

17) $\dfrac{x^2 + 2x - 3}{x^2 + 5x + 6} = \dfrac{x - 1}{x + 2}$

18) $\dfrac{9x^3 + 53x^2 - 9x - 18}{x^2 + 11x + 30} = \dfrac{9x^2 - x - 3}{x + 5}$

19) $\dfrac{2x^3 + x^2 - 8x + 5}{7x^2 - 12x + 5} = \dfrac{2x^2 + 3x - 5}{7x - 5}$

20) $$\frac{2x^3+3x^2+x}{x^3-x^2-2x}=\frac{2x+1}{x-2}$$

21) $$\frac{a^3b^3+c^3x^3}{a^2b^2-c^2x^2}=\frac{a^2b^2-abcx+c^2x^2}{ab-cx}$$

22) $$\frac{ax^m-bx^{m+1}}{a^2bx-b^3x^3}=\frac{x^{m-1}}{ab+b^2x}$$

23) $$\frac{2x^3-(3c+d+2)x^2+(3c+d)x}{x^4-x}=\frac{2x-3c-d}{x^2+x+1}$$

24) $$\frac{a^2+b^2+c^2+2ab+2ac+2bc}{a^2-b^2-c^2-2bc}=\frac{a+b+c}{a-b-c}$$

25) $$\frac{a^2-3ab+ac+2b^2-2bc}{a^2-b^2+2bc-c^2}=\frac{a-2b}{a+b-c}$$

26) $$\frac{(a+b)(a+b+c)(a+b-c)}{2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4}=$$

$$\frac{(a+b)(a+b+c)(a+b-c)}{4b^2c^2-(a^2-b^2-c^2)^2}=\frac{a+b}{(c+a-b)(b-a+c)}{}^{*}$$

---

3) MIXED REDUCTIONS.

1) $$\sqrt{ax}+\frac{ax}{a-\sqrt{ax}}=\frac{a\sqrt{ax}}{a-\sqrt{ax}}=\frac{a\sqrt{x}}{\sqrt{a}-\sqrt{x}}$$

2) $$\frac{c\sqrt{x}}{\sqrt{(a+x)}}+\frac{d\sqrt{x}}{\sqrt{(a-x)}}+\frac{a\sqrt{(ax^3+x^4)}}{\sqrt{(a^2-x^2)}}-\sqrt{(a^2-x^2)}$$
$$=\frac{c\sqrt{(ax-x^2)}+(ax+d)\sqrt{(ax+x^2)}+x^2-a^2}{\sqrt{(a^2-x^2)}}$$

---

* Upon this reduction the teacher has an opportunity to make several important remarks.

3) $$\frac{2x^2}{(1-x^2)^{\frac{3}{2}}}-\frac{1}{(1-x^2)^{\frac{1}{2}}}=\frac{3x^2-1}{(1-x^2)\sqrt{(1-x^2)}}$$

4) $$\frac{ax^3}{(a+x)^{\frac{8}{3}}}+\frac{bx^2}{(a+x)^{\frac{5}{3}}}+\frac{cx}{(a+x)^{\frac{2}{3}}}=$$
$$\frac{(a+b+c)x^3+(ab+2ac)x^2+a^2cx}{(a+x)^2\sqrt[3]{(a+x)^2}}$$

5) $$\sqrt{\left(1-\frac{f^2}{(f-g)^2}\right)}=\frac{\sqrt{(g^2-2fg)}}{f-g}$$

6) $$\frac{a+\sqrt{-b}}{a-\sqrt{-b}}=\frac{a^2-b+2a\sqrt{-b}}{a^2+b}$$

7) $$\frac{a+\sqrt{-b}}{a-\sqrt{-b}}+\frac{a-\sqrt{-b}}{a+\sqrt{-b}}=\frac{2(a^2-b)}{a^2+b}$$

8) $$\frac{\sqrt{(a+x)}+\sqrt{(a-x)}}{\sqrt{(a+x)}-\sqrt{(a-x)}}=\frac{a+\sqrt{(a^2-x^2)}}{x}$$

9) $$\frac{b}{\sqrt[n]{[a-\sqrt{(a^2-b^n)}]}}=\sqrt[n]{[a+\sqrt{(a^2-b^n)}]}$$

10) $$\sqrt{(a+\sqrt{b})}\pm\sqrt{(a-\sqrt{b})}=\sqrt{[2a\pm2\sqrt{(a^2-b)}]}\,*$$

11) $$\sqrt{(a+\sqrt{-b})}\pm\sqrt{(a-\sqrt{-b}}=\sqrt{[2a\pm2\sqrt{(a^2+b)}]}$$

12) $$\sqrt{\left(\frac{abf+c^2}{bc}+\sqrt{\frac{4af}{b}}\right)}+\sqrt{\left(\frac{abf+c^2}{bc}+\sqrt{\frac{4af}{b}}\right)}$$
$$=\sqrt{\frac{4af}{c}}$$

---

* This reduction, and the two following ones, may be effected two ways: 1st, by extracting the root of $a+\sqrt{b}$, as well as of $a-\sqrt{b}$, and adding them; or 2dly, by raising the whole expression to the square, and taking the root of it.

13) $$\frac{\frac{a}{b}+\frac{c}{d}}{\frac{e}{f}+\frac{g}{h}}=\frac{(ad+bc)fh}{(eh+fg)bd}$$

14) $$\frac{\frac{a}{b}+\frac{c}{d}+\frac{e}{f}}{\frac{g}{h}+\frac{i}{k}+\frac{l}{m}}=\frac{(adf+bcf+bde)hkm}{(gkm+him+hkl)bdf}$$

15) $$\frac{\frac{a^3f^3}{b^2c^2}-\frac{a^4f}{bc}+a^2c}{\frac{a^2g}{bc^2d}-\frac{a^6c}{b^2g^2h}+\frac{a^3}{bc}}=\frac{(af^3-a^2bcf+b^2c^3)dg^2h}{bg^3h-a^4c^3d+abcdg^2h}$$

16) $$\frac{\frac{a}{a-b}+\frac{b}{a+b}}{\frac{a}{a-b}-\frac{b}{a+b}}=\frac{a^2+2ab-b^2}{a^2+b^2}$$

17) $$\frac{\frac{c^2}{d^2}-\frac{c^3}{a+b}}{\frac{c^2}{a+b}-\frac{c^4}{dh^2}}=\frac{(a+b-cd^2)h^2}{d^2h^2-(a+b)c^2d}$$

18) $$\frac{1+\frac{\sqrt{(a^2-x^2)}}{\sqrt{(a^2+x^2)}}}{\sqrt{(a^2+x^2)}+\sqrt{(a^2-x^2)}}=\frac{1}{\sqrt{(a^2+x^2)}}$$

19) $$\frac{\sqrt{(1-x)}+\frac{1}{\sqrt{(1+x)}}}{1+\frac{1}{\sqrt{(1-x^2)}}}=\sqrt{(1-x)}$$

20) $$\frac{a^2+ax+x^2}{a^4+a^3x+a^2x^2+ax^3+x^4}=\frac{a^3-x^3}{a^5-x^5}$$

21) $\frac{a^3 - a^2x - ax^2 - x^3}{a^5 - a^4x + a^3x^2 - a^2x^3 + ax^4 - x^5} = \frac{a^4 - x^4}{a^6 - x^6}$

22) $\frac{a^2 - 2ax + 4x^2}{a^3 - 2a^2x + 4ax^2 - 8x^3} = \frac{a^3 + 8x^3}{a^4 - 16x^4}$

23) $9\sqrt{(6\sqrt{28})} + 3\sqrt{(12\sqrt{7})} - 8\sqrt{(4\sqrt{63})} = 8\sqrt[4]{63}$

24) $3\sqrt{(40\sqrt{12})} + 2\sqrt{(5\sqrt{48})} - 4\sqrt{(15\sqrt{27})} = 4\sqrt[4]{75}$

25) $4\sqrt[3]{(6\sqrt{32})} + \sqrt[3]{(9\sqrt{162})} + 2\sqrt[3]{(75\sqrt{50})} = 21\sqrt[6]{18}$

26) $5\sqrt[3]{(4\sqrt[3]{192}} + 7\sqrt[3]{(18\sqrt[3]{81})} - 31\sqrt[9]{24}$

27) $3\sqrt[3]{(8 + 16\sqrt{5})} - 2\sqrt[3]{(1 + \sqrt{20})} = 4\sqrt[3]{(1 + 2\sqrt{5})}$

28) $3\sqrt[3]{(54 - 36\sqrt{27})} - \sqrt[3]{(16 - 16\sqrt{12})} = 7\sqrt[3]{(2 - 4\sqrt{3})}$

29) $(aa' + bb')^2 + (ab' - ba')^2 = (a^2 + b^2)(a'^2 + b'^2)$ *

30) $(aa' + bb')^2 + (ab' - ba')^2 + a'^2c^2 + b'^2c^2 =$
$(a^2 + b^2 + c^2)(a'^2 + b'^2)$

31) $(aa' + bb' + cc')^2 + (ab' - ba')^2 + (ac' - ca')^2$
$+ (bc' - cb')^2 = (a^2 + b^2 + c^2)(a'^2 + b'^2 + c'^2)$

32) $(aa' + bb' + cc' + dd')^2 + (ab' - ba' + cd' - dc')^2$
$+ (ac' - bd' - ca' + db')^2 + (ad' + bc' - cb' - da')^2$
$= (a^2 + b^2 + c^2 + d^2)(a'^2 + b'^2 + c'^2 + d'^2)$

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* For the sake of symmetry, we frequently denote quantities by letters with an accent or dash over them; but then the letters a, b, c, &c; a', b', c', &c; a'', b'', c'', &c; a''', b''', c''', &c, denote quantities which are distinct from each other.

33) $(a^2 + Ab^2)\ (a'^2 + Ab'^2) = (aa' \pm Abb')^2 + A(ab' \mp ba')^2$ *

34) $(ab' - ba')\ (ab'' - ba'') + (bc' - cb')\ (bc'' - cb'') + (ca' - ac')\ (ca'' - ac'') = (a^2 + b^2 + c^2)\ (a'a'' + b'b'' + c'c'') - (aa' + bb' + cc')\ (aa'' + bb'' + cc'')$

XI. LOGARITHMS.

What is understood by the logarithm of a number ? What by its basis ?—What are we to understand when it is said, for the basis a, the logarithm of a number N, is 6,67 ? What is a logarithmic system ? What, particularly, is Brigg's system ? How are the three principal formulæ, which follow hereafter, expressed in words ? How can they be demonstrated ? Can unity be taken for the basis of any logarithmic system ? What is the logarithm of unity in every system ? — When the basis of a logarithmic system is greater than unity, then the logarithms of all numbers greater than 1, are positive ; whereas, the logarithms of all proper fractions are nega-

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* The formulæ 29, 30, 31, 32, 33, solve several problems in Indeterminate Analysis, as we shall see hereafter. Their correctness may be proved by the actual developement of the squares and products. In this developement several reductions may be made, which the pupil himself may discover, by paying some attention to the nature of the expression.

tive; but how is it, when the basis is less than unity (a proper fraction)? A few logarithms only are integers; the others are a whole number together with an imperfect fraction, that is, one which cannot be exactly expressed. What is the name of the whole number, and what that of the fraction? What is, in Brigg's system, the index of a number between 10^n and 10^{n+1}, and what is the index of a fraction between $\frac{1}{10^n}$ and $\frac{1}{10^{n+1}}$?

When the differences of numbers are small, compared with the numbers themselves, then the differences of their logarithms are nearly in proportion to the numbers. The reason why this is so is explained in Analysis. What is the use of the proportional parts, which are found in the larger tables of logarithms?

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1) PRINCIPAL FORMULÆ.

1) $\log AB = \log A + \log B$

2) $\log \frac{A}{B} = \log A - \log B$

3) $\log A^n = n \log A$

Remark. In 3) n may be positive or negative, a whole or a fractional number.

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2) APPLICATION OF THESE FORMULÆ TO THE FINDING OF LOGARITHMS OF PRODUCTS, QUOTIENTS, POWERS AND ROOTS.

a. FOR GENERAL OR ALGEBRAIC EXPRESSIONS.

1) $\log abcd = \log a + \log b + \log c + \log d$

2) $\log \frac{fg}{cd} = \log f + \log g - \log c - \log d$

3) $\log a^m b^n c^p = m \log a + n \log b - p \log c$

4) $\log \frac{a^m b^{-n}}{c^p d^q} = m \log a - n \log b - p \log c - q \log d$

5) $\log a^{\frac{m}{n}} b^{-\frac{p}{q}} c = \frac{m}{n} \log a - \frac{p}{q} \log b + \log c$

6) $\log \sqrt[n]{a^m b^{-n} c^{\frac{p}{q}}} = \frac{m}{n} \log a - \log b + \frac{p}{nq} \log c$

7) $\log \frac{a\sqrt[n]{c^m}}{b\sqrt{d}} = \log a + \frac{m}{n} \log c - \log b - \frac{1}{2} \log d$

8) $\log \frac{(a+b)^n c^m}{(c+d)\sqrt[4]{d^3}} = n \log (a+b) + m \log c - \log (c+d)$

$- \frac{3}{4} \log d$

9) $\log \frac{1}{(a+b^n)^m} = - m \log (a + b^n)$

10) $\log \frac{1}{\sqrt[n]{(a+b)}} = - \frac{1}{n} \log (a+b)$

11) $\log \sqrt[m]{(a^2 - x^2)} = \frac{1}{m} \log (a^2 - x^2) = \frac{1}{m} \log (a + x)$

$+ \frac{1}{m} \log (a - x)$

12) $x \log a = \log a^x$

13) $n \log a + m \log b - p \log c = \log \frac{a^n b^m}{c^p}$

14) $n \log(a+y) + \log c - m \log(a-y) = \log \frac{c(a+y)^n}{(a-y)^m}$

15) $\frac{1}{n} \log(2a+3b) - \frac{2}{3} \log c = \log \frac{\sqrt[n]{(2a+3b)}}{\sqrt[3]{c^2}}$

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b. FOR NUMERICAL EXPRESSIONS, IN BRIGG'S SYSTEM.

1) $\log(93 \times 3514) = 5,5142847$

2) $\log(1225 \times 387) = 5,6758471$

3) $\log(628 \times 493) = 5,4908066$

4) $\log(3748 \times 1752 \times 4065) = 10,4263942$

5) $\log \frac{5}{4} = 0,0969100$

6) $\log \frac{39}{7} = 0,7459666$

7) $\log \frac{14}{3} = 0,6690068$

8) $\log 15\frac{3}{4} = 1,1972806$

9) $\log 7\frac{4}{13} = 0,8637803$

10) $\log 367\frac{5}{8} = 2,5654050$

11) $\log 187\frac{9}{11} = 2,2737376$

12) $\log \frac{2}{3} = 0,8239087 - 1$

13) $\log \frac{5}{8} = 0,7958800 - 1$

14) $\log \frac{1}{9} = 0,0457575 - 1$

15) $\log \frac{1}{26} = 0{,}5850267 - 2$

16) $\log \frac{6}{37} = 0{,}2099495 - 1$

17) $\log \frac{18}{1267} = 0{,}1524959 - 2$

18) $\log \frac{357}{695} = 0{,}7106834 - 1$

19) $\log \frac{1}{5432} = 0{,}2650402 - 4$

20) $\log \frac{8}{9243} = 0{,}9372770 - 4$

21) $\log \frac{3542}{7965} = 0{,}6480628 - 1$

22) $\log 3{,}5 = 0{,}5440680$

23) $\log 12{,}63 = 1{,}1014034$

34) $\log 15{,}432 = 1{,}1884222$

25) $\log 7348{,}4 = 3{,}8661928$

26) $\log 1{,}3567 = 0{,}1324838$

27) $\log 0{,}7 = 0{,}8450980 - 1$

28) $\log 0{,}036 = 0{,}5563025 - 2$

29) $\log 0{,}0065 = 0{,}\ 8129134 - 3$

30) $\log 0{,}0039953 = 0{,}6015494 - 3$

31) $\log 0{,}0005637 = 0{,}7510480 - 4$

32) $\log \frac{319 \times 765}{138} = 3{,}2475730$

33) $\log \frac{213 \times 7{,}655}{3145 \times 718} = 0{,}8585798 - 4$

34) $\log \frac{3{,}5347 \times 2{,}685}{137{,}65 \times 5944} = 0{,}0644419 - 5$

35) $\log \frac{47 \times 0{,}653 \times 12\frac{5}{6}}{3576 \times 1520} = 0{,}8601095 - 5$

36) $\log \frac{0{,}765 \times 0{,}0018}{31457 \times 567\frac{5}{12}} = 0{,}8873146 - 11$

37) $\log \dfrac{0{,}018594 \times 763\frac{13}{15}}{7654{,}3 \times 794} = 0{,}3686643 - 6$

38) $\log 3^{15} = 7{,}1568188$ *

39) $\log 5^{27} = 18{,}8721901$

40) $\log 16^{20} = 24{,}0823997$

41) $\log (\frac{7}{3})^{14} = 5{,}1516750$

42) $\log (\frac{15}{13})^{16} = 0{,}9943665$

43) $\log (\frac{12}{5})^{32} = 12{,}1667597$

44) $\log (\frac{3}{4})^{30} = 0{,}2518379 - 4$

45) $\log (\frac{357}{1432})^{12} = 0{,}7607024 - 8$

46) $\log (\frac{1}{14})^{125} = 0{,}7339955 - 144$

47) $\log [(14{,}418)^{9} \times (3{,}71)^{6}] = 13{,}8463886$

48) $\log [(0{,}0534)^{3} \times (\frac{32768}{3875})^{10}] = 5{,}4544061$

49) $\log \dfrac{(0{,}5936)^{20} \times 386}{(0{,}076)^{23}} = 23{,}7977525$

50) $\log \sqrt{5} = 0{,}3494850$

51) $\log \sqrt{73567} = 2{,}4333415$

52) $\log \sqrt[3]{135} = 0{,}7101112$

53) $\log \sqrt[8]{15276} = 0{,}5230012$

54) $\log \sqrt[5]{35107} = 0{,}9090787$

55) $\log \sqrt[100]{13} = 0{,}0111394$

---

* To obtain the logarithms of higher powers, correct to 7 decimals, it is necessary to use tables calculated to *more* than 7 decimals; otherwise the last figure in the result will not agree with the above

56) $\log \sqrt[7]{\frac{15}{4}} = 0{,}0820045$

57) $\log \sqrt[5]{\frac{4}{9}} = 0{,}9295635 - 1$

58) $\log \sqrt[16]{\frac{3587}{20593}} = 0{,}9525632 - 1$

59) $\log \sqrt[35]{\frac{803}{91056}} = 0{,}9412973 - 1$

60) $\log \sqrt[17]{(954)^{12}} = 2{,}1032106$

61) $\log \sqrt[11]{(\frac{12}{7})^{28}} = 0{,}5958482$

62) $\log \sqrt[13]{(\frac{547}{935})^{207}} = 0{,}2927210 - 4$

63) $\log \sqrt[14]{(\frac{1}{3})^{187}} = 0{,}6270232 - 7$

64) $\log \sqrt[16]{(\frac{14}{267})^{715}} = 0{,}7828746 - 58$

65) $\log \sqrt[80]{0{,}00534} = 0{,}9715943 - 1$

66) $\log \sqrt[540]{0{,}00007} = 0{,}9923057 - 1$

67) $\log \sqrt[12]{(0{,}34576)^{7}} = 0{,}7309519 - 1$

68) $\log \sqrt[32]{(356{,}27)^{11}} = 0{,}8771741$

69) $\log \sqrt[5]{\frac{0{,}365 \times \sqrt{2}}{788}} = 0{,}3632563 - 1$

70) $\log \sqrt[10]{\frac{78563\sqrt[3]{\frac{5}{3}}}{15\sqrt[4]{0{,}2}}} = 0{,}3967819$

71) $\log \sqrt[9]{\frac{347\sqrt[7]{0{,}0073}}{126\sqrt[3]{\frac{4}{9}}}} = 0{,}0280126$

3) USE OF PROPORTIONAL PARTS OF LOGARITHMS.

a. FINDING THE LOGARITHMS OF SUCH NUMBERS AS EXCEED THE LIMITS OF THE TABLES.

1) $\log 1851273 = 6{,}2674705$
2) $\log 14459809 = 7{,}1601626$
3) $\log 10134761 = 7{,}0058135$
4) $\log 7095137 = 6{,}8509608$
5) $\log 506860900 = 8{,}7048888$
6) $\log 3{,}614699 = 0{,}5580721$
7) $\log 84{,}827567 = 1{,}9285370$
8) $\log 211447{,}39 = 5{,}3252023$
9) $\log 0{,}0013514133 = 0{,}1307882 - 3$
10) $\log 0{,}0003599547 = 0{,}5562478 - 4$
11) $\log 75907\frac{1}{8} = 4{,}8802825$
12) $\log 32116\frac{7}{9} = 4{,}5067320$
13) $\log 2528811\frac{1}{4} = 6{,}4029164$
14) $\log 522076\frac{2}{13} = 5{,}7177339$

b. FINDING OF NUMBERS WHICH BELONG TO SUCH LOGARITHMS AS ARE NOT EXACTLY TO BE FOUND IN THE TABLES.

1) num. log 1,0742664 = 11,86496....
2) num. log 3,5947835 = 3933,538....
3) num. log 0,7813427 = 6,044254....
4) num. log 2,0037683 = 100,8714....
5) num. log 4,0005673 = 10013,07....
6) num. log 5,6165834 = 413602,7.....
7) num. log 3,7694480 = 5880,956....
8) num. log 0,2307611 = 1,701222....
9) num. log 4,2933065 = 19602,27....
10) num. log 6,1785400 = 1508481,....

---

4) ACTUAL CALCULATION OF SOME NUMERICAL EXPRESSIONS BY MEANS OF LOGARITHMS.

1) $\sqrt[7]{8} = 1{,}345900....$
2) $\sqrt[4]{35246} = 13{,}70179....$
3) $\sqrt[12]{567348} = 3{,}016389....$
4) $\sqrt[6]{235{,}78} = 2{,}485522....$
5) $\sqrt[5]{\frac{13}{16}} = 0{,}959322....$

6) $\sqrt[7]{\frac{1171}{345}} = 1{,}190747....$

7) $\sqrt[3]{17705\frac{2}{9}} = 26{,}06356....$

8) $\sqrt[9]{1350\frac{7}{8}} = 2{,}227645....$

9) $\sqrt[8]{172\frac{5}{6}} = 1{,}904159....$

10) $\sqrt[13]{\frac{3348}{569}} = 1{,}146055....$

11) $(\frac{9}{8})^{21} = 11{,}86322....$

12) $(2\frac{5}{6})^{9} = 11767{,}35....$

13) $(\frac{643}{637})^{123} = 3{,}168104....$

14) $(317\frac{3}{4})^{0{,}6} = 31{,}71402....$

15) $(\frac{167}{53})^{0{,}32} = 1{,}443779....$

16) $(\frac{5}{7})^{0{,}0537} = 0{,}982093....$

17) $\frac{(991{,}767)^5 \times 12{,}34}{(20{,}358 \times 10{,}1575)^6} = 151{,}5369....$

18) $\frac{(52072)^{13} \times \sqrt{(0{,}000734)^9}}{(255608)^8} = 8930{,}834....$

19) $\left(\frac{42666}{1147}\right)^{12} \times \left(\frac{765}{19432}\right)^{10} = 62756{,}88....$

20) $\sqrt[5]{(\frac{7}{3}\sqrt[4]{6})} = 1{,}295695....$

21) $\sqrt[3]{(0{,}26 \cdot \sqrt{\frac{2}{3}})} = 0{,}596544....$

22) $\sqrt[5]{\frac{3425\sqrt[7]{136}}{0{,}00034}} = 28{,}94639....$

23) $253\sqrt[3]{\frac{716{,}5}{\sqrt{2}}} = 2016{,}914....$

24) $\sqrt[4]{\frac{132 \times (7{,}356)^9}{\sqrt{(3{,}25)^5}}} = 144{,}5972....$

25) $$\frac{\sqrt[7]{(466871)^6} \times \sqrt[9]{(3576)^{16}}}{996003\sqrt{0,0071}} = 1788845,....$$

26) $\sqrt[8]{(21 + \sqrt[6]{19})} = 1,476875....$

27) $\sqrt[3]{(5,03 + \sqrt[5]{0,2})} = 1,792020....$

28) $\sqrt[5]{(9,921 - 3\sqrt{5,02})} = 1,261866....$

29) $$\sqrt[16]{\frac{43 + 5\sqrt[3]{278}}{\sqrt[5]{17}}} = 1,264848....$$

The following theorems are yet deserving of our notice.

1) Let *A*, *B*, denote two logarithmic systems, whose respective bases are *a*, *b*. Let also x, y, represent the logarithms of the same number *k*, in these systems. Then we have always $y : x = \log a : \log b$; where the two last logarithms are taken in any third system whatever.

2) From this proportion it follows, that the logarithm of a number *k*, in the system *B*, is found, by dividing the logarithm of the number *k*, in the system *A*, by the logarithm of the basis *b*, taken in the same system *A*.

3) The logarithms of the same numbers, in two different systems, have a constant ratio.

4) If, therefore, we know the logarithm of one and the same number in two different systems, we can always find a number by which we may multiply the logarithms in one system, in order to find the logarithms of the same numbers in the other. This number is called the *Modulus*. (In the common acceptation of the word, it means only that number, by which we must multiply the logarithms in the hyper-

bolic, or natural system, whose basis is 2,718281828459.... in order to obtain the logarithms of the same number in Brigg's, or any other system.

X. PERMUTATIONS, COMBINATIONS, AND VARIATIONS.*

What is a permutation? What a combination? What a variation? In what respects do they differ from each other? What rules are observed? What rules are observed in permutations and combinations?

Complete works on the theory of Combinations, are "Stahl's Elements of the Theory of Combinations, with its Application to Analysis."—Leipzig, 1800; and "Weingaertner's Treatise on Combinatory Analysis, according to the theory of Professor Hindenburg."—Leipzig, 1801. (Stahl's Grundriss der Combinationslehre, nebst Anwendung derselben auf die Analysis. Leipzig, 1800. Weingaertner's Lehrbuch der Kombinatorischen Analysis nach der Theorie des Herrn. Prof. Hindenburg. Leipzig, 1801.)

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* *Permutation* means the transposition that can be effected, among any number of given things, all of them being taken together. By *variation* we understand the transpositions that can take place when the things are *not all* taken at once, but only 2, 3, 4, 5, &c, of them at a time. English and French writers generally consider both cases as one, and call both Permutations.

1) PERMUTATIONS.

a. ACTUAL VIEW OF THE PERMUTATIONS OF A GIVEN NUMBER OF THINGS OR ELEMENTS.

1) Perm. (*abc*) =

abc
acb
bac
bca
cab
cba

2) Perm. (*aabb*) =

aabb
abab
abba
baab
baba
bbaa

3) Perm. (*abcd*) =

abcd
abdc
acbd
acdb
adbc
adcb
bacd
badc
bcad
bcda
bdac
bdca

cabd
cadb
cbad
cbda
cdab
cdba
dabc
dacb
dbac
dbca
dcab
dcba

4) Perm. (*abbc*) =

abbc
abcb
acbb
babc
bacb
bbac
bbca
bcab
bcba
cabb
cbab
cbba

5) Perm. $\underbrace{(aabbc) =}$

aabbc
aabcb
aacbb
ababc
abacb
abbac
abbca
abcab
abcba
acabb
acbab
acbba
baabc
baacb
babac
babca
bacab
bacba
bbaac
bbaca
bbcaa
bcaab
bcaba
bcbaa
caabb
cabab
cabba
cbaab
cbaba
cbbaa

6) Perm. $(aaaabb) =$

aaaabb
aaabab
aaabba
aabaab
aababa
aabbaa
abaaab
abaaba
ababaa
abbaaa
baaaab
baaaba
baabaa
babaaa
bbaaaa

7) Perm. $\underbrace{(aabcd) =}$

aabcd
aabdc
aacbd
aacdb
aadbc
aadcb
abacd
abadc
abcad
abcda
abdac
abdca
acabd
acadb
acbad
acbda
acdab
acdba
adabc
adacb
adbac
adbca
adcab
adcba
baacd
baadc
bacad
bacda
badac
badca
bcaad
bcada
bcdaa
bdaac
bdaca
bdcaa
caabd
caadb
cabad
cabda
cadab
cadba
cbaad
cbada
cbdaa
cdaab
cdaba
cdbaa
daabc
daacb
dabac
dabca
dacab
dacba
dbaac
dbaca
dbcaa
dcaab
dcaba
dcbaa

8) Perm. (*aabbcc*) =

aabbcc
aabcbc
aabccb
aacbbc
aacbcb
aaccbb
ababcc
abacbc
abaccb
abbacc
abbcac
abbcca
abcabc
abcacb
abcbac
abcbca
abccab
abccba
acabbc
acabcb
acacbb
acbabc
acbacb
acbbac
acbbca
acbcab
acbcba
accabb
accbab
accbba
baabcc
baacbc
baaccb
babacc
babcac
babcca
bacabc
bacacb
bacbac
bacbca
baccab
baccba
bbaacc
bbacac
bbacca
bbcaac
bbcaca
bbccaa
bcaabc
bcaacb
bcabac
bcabca
bcacab
bcacba
bcbaac
bcbaca
bcbcaa
bccaab
bccaba
bccbaa
caabbc
caabcb
caacbb
cababc
cabacb
cabbac
cabbca
cabcab
cabcba
cacabb
cacbab
cacbba
cbaabc
cbaacb
cbabac
cbabca
cbacab
cbacba
cbbaac
cbbaca
cbbcaa
cbcaab
cbcaba
cbcbaa
ccaabb
ccabab
ccabba
ccbaab
ccbaba
ccbbaa

b. NUMBER OF POSSIBLE PERMUTATIONS.[*]

FORMULÆ.

I. The number of possible permutations of N different things is $= 1 . 2 . 3 . 4 . 5 \ldots\ldots\ldots (N - 1) N$.

II. If among the N given things there are l of one kind,

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[*] The exponents which occur here are mere exponents of repetition, and denote how often the letter to which they belong is repeated.—n. P. signifies *Numerus Permutationum* (number of transpositions of permutations).

m of another, *n* of a third, &c, then the number of permutations, which will actually differ from each other, is

$$= \frac{1.2.3.4 \ldots\ldots\ldots\ldots (N-1)\,N}{1.2.3 \ldots l \times 1.2.3 \ldots m \times 1.2.3 \ldots n \times 1.2.3 \ldots p \times \ldots}$$

$$= \frac{(l+1)(l+2)(l+3) \ldots\ldots\ldots (N-1)N}{1.2.3 \ldots m \times 1.2.3 \ldots n \times 1.2.3 \ldots p \times \ldots}$$

$$= \frac{(m+1)(m+2)(m+3) \ldots\ldots\ldots (N-1)N}{1.2.3 \ldots l \times 1.2.3 \ldots n \times 1.2.3 \ldots p \times \ldots}$$

$$= \frac{(n+1)(n+2)(n+3) \ldots\ldots\ldots (N-1)N}{1.2.3 \ldots l \times 1.2.3 \ldots m \times 1.2.3 \ldots p \times \ldots}$$

where there are as many different products 1.2.3.4 ... *l*, 1.2.3 ... *m*, 1.2.3 ... *n*, 1.2.3 ... *p*, &c, as there are collections of equal elements among the given number *N*.

EXAMPLES.

1) n. P. $(a) = 1$
2) n. P. $(ab) = 2$
3) n. P. $(abc) = 6$
4) n. P. $(abcd) = 24$
5) n. P. $(abcde) = 120$
6) n. P. $(abcdef) = 720$
7) n. P. $(a^2bc) = 12$
8) n. P. $(a^2b^2) = 6$
9) n. P. $(a^3b^5c^2) = 2520$
10) n. P. $(a^2b^4cd) = 840$
11) n. P. $(a^2b^2c^2) = 90$
12) n. P. $(a^3b^7c^4d) = 1801800$
13) n. P. $(a^2b^3c^8d^4) = 30630600$

14) n. P. $(a^2b^{10}cd^2e^3) = 73513440$

15) n. P. $(a^5b^5c^4d^2) = 30270240$

16) n. P. $(a^{13}b^2c^5d^3e^2) = 864913896000$

7) n. P. $(a^3b^7c^4d^2e^3f^5) = 593676898214400$

18) n. P. $(a^m) = 1$

19) n. P. $(a^{m-1}b) = m$

20) n. P. $(a^{m-2}b^2) = \frac{m(m-1)}{1\,.\,2}$

21) n. P. $(a^{m-3}b^3) = \frac{m(m-1)\,(m-2)}{1\,.\,2\,.\,3}$

22) n. P. $(a^{m-4}b^4) = \frac{m(m-1)\,(m-2)\,(m-3}{1\,.\,2\,.\,3\,.\,4}$

23) n. P. $(a^{m-5}b^5) = \frac{m(m-1)\,(m-2)\,(m-3)\,(m-4)}{1\,.\,2\,.\,3\,.\,4\,.\,5}$

and so on.

2) COMBINATIONS.

a. COMBINATIONS WITH REPETITIONS.

1) Comb. (a, b, c, d) of the second Class.

aa, ab, ac, ad, bb, bc, bd, cc, cd, dd

2) Comb. (a, b, c, d, e) of the second Class.

aa, ab, ac, ad, ae, bb, bc, bd, be, cc, cd, ce, dd, de, ee,

3) Comb. (*a*, *b*, *c*)
(Third Class.)

aaa
aab
aac
abb
abc
acc
bbb
bbc
bcc
ccc

4) Comb. *a*, *b*, *c*, *d*)
(Third Class.)

aaa
aab
aac
aad
abb
abc
abd
acc
acd
add

bbb
bbc
bbd
bcc
bcd
bdd
ccc
ccd
cdd
ddd

5) Comb. (*a*, *b*, *c*, *d*, *e*)
(Third Class.)

aaa
aab
aac
aad
aae
abb
abc
abd
abe
acc
acd
ace

add
ade
aee
bbb
bbc
bbd
bbe
bcc
bcd
bce
bdd
bde

bee
ccc
ccd
cce
cdd
cde
cee
ddd
dde
dec
eee

6) Comb. *a*, *b*, *c*)
(Fourth Class.)

aaaa
aaab
aaac
aabb
aabc
aacc
abbb
abbc
abcc
accc
bbbb
bbbc
bbcc
bccc
cccc

7) Comb. (a, b, c, d)
(Fourth Class.)

aaaa
aaab
aaac
aaad
aabb
aabc
aabd
aacc
aacd
aadd
abbb
abbc
abbd
abcc
abcd
abdd
accc
accd
acdd
addd
bbbb
bbbc
bbbd
bbcc
bbcd
bbdd
bccc
bccd
bcdd
bddd
cccc
cccd
ccdd
cddd
dddd

8) Comb. (a, b, c, d, e)
(Fourth Class.)

aaaa
aaab
aaac
aaad
aaae
aabb
aabc
aabd
aabe
aacc
aacd
aace
aadd
aade
aaee
abbb
abbc
abbd
abbe
abcc
abcd
abce
abdd
abde
abee
accc
accd
acce
acdd
acde
acee
addd
adde
adee
aeee
bbbb
bbbc
bbbd
bbbe
bbcc
bbcd
bbce
bbdd
bbde
bbee
bccc
bccd
bcce
bcdd
bcde
bcee
bddd
bdde
bdee
beee
cccc
cccd
ccce
ccdd
ccde
ccee
cddd
cdde
cdee
ceee
dddd
ddde
ddee
deee
eeee

9) Comb. (a, b, c)
(Fifth Class.)

aaaaa
aaaab
aaaac
aaabb
aaabc
aaacc
aabbb
aabbc
aabcc
aaccc
abbbb
abbbc
abbcc
abccc
acccc
bbbbb
bbbbc
bbbcc
bbccc
bcccc
ccccc

10) Comb. (a, b, c, d)
(Fifth Class.)

aaaaa
aaaab
aaaac
aaaad
aaabb
aaabc
aaabd
aaacc
aaacd
aaadd
aabbb
aabbc
aabbd
aabcc
aabcd
aabdd
aaccc
aaccd
aacdd
aaddd
abbbb
abbbc
abbbd
abbcc
abbcd
abbdd
abccc
abccd
abcdd
abddd
acccc
acccd
accdd
acddd
adddd
bbbbb
bbbbc
bbbbd
bbbcc
bbbcd
bbbdd
bbccc
bbccd
bbcdd
bbddd
bcccc
bcccd
bccdd
bcddd
bdddd
ccccc
ccccd
cccdd
ccddd
cdddd
ddddd

The number of combinations with repetitions for n Elements is

For the 1st Class $= n$

— — 2d Class $= \frac{n(n+1)}{1 \,.\, 2}$

— — 3d Class $= \frac{n(n+1)\,(n+2)}{1 \,.\, 2 \,.\, 3}$

— — 4th Class $= \frac{n(n+1)\,(n+2)\,(n+3)}{1 \,.\, 2 \,.\, 3 \,.\, 4}$

. . . .

— — mth Class $= \frac{n(n+1)\,(n+2)\ldots\ldots(n+m-1)}{1 \,.\, 2 \,.\, 3 \ldots\ldots\ldots\ldots m}$

---

b. COMBINATIONS WITHOUT REPETITIONS.

1) Comb. (*a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, *i*)
(Second Class.)

ab
ac
ad
ae
af
ag
ah
ai
bc
bd
be
bf
bg
bh
bi
cd
ce
cf
cg
ch
ci
de
df
dg
dh
di
ef
eg
eh
ei
fg
fh
fi
gh
gi
hi

2) Comb. *a*, *b*, *c*, *d*, *e*, *f*)
(Third Class.)

abc
abd
abe
abf
acd
ace
acf
ade
adf
aef
bcd
bce
bcf
bde
bdf
bef
cde
cdf
cef
def

3) Comb. (a, b, c, d, e, f, g, h)
(Third Class.)

abc
abd
abe
abf
abg
abh
acd
ace
acf
acg
ach
ade
adf
adg
adh
aef
aeg
aeh
afg
afh
agh
bcd
bce
bcf
bcg
bch
bde
bdf
bdg
bdh
bef
beg
beh
bfg
bfh
bgh
cde
cdf
cdg
cdh
cef
ceg
ceh
cfg
cfh
cgh
def
deg
deh
dfg
dfh
dgh
efg
efh
egh
fgh

3) Comb. (a, b, c, d, e, f, g)
(Fourth Class.)

abcd
abce
abcf
abcg
abde
abdf
abdg
abef
abeg
abfg
acde
acdf
acdg
acef
aceg
acfg
adef
adeg
adfg
aefg
bcde
bcdf
bcdg
bcef
bceg
bcfg
bdef
bdeg
bdfg
befg
cdef
cdeg
cdfg
cefg
defg

5) Comb. (a, b, c, d, e, f, g, h)

(Fifth Class.)

abcde
abcdf
abcdg
abcdh
abcef
abceg
abceh
abcfg
abcfh
abcgh
abdef
abdeg
abdeh
abdfg
abdfh
abdgh
abefg
abefh
abegh
abfgh
acdef
acdeg
acdeh
acdfg
acdfh
acdgh
acefg
acefh
acegh
acfgh
adefg
adefh
adegh
adfgh
aefgh
bcdef
bcdeg
bcdeh
bcdfg
bcdfh
bcdgh
bcefg
bcefh
bcegh
bcfgh
bdefg
bdefh
bdegh
bdfgh
befgh
cdefg
cdefh
cdegh
cdfgh
cefgh
defgh

The number of Combinations without repetitions for n Elements, is

For the 1st Class $= n$

— — 2d Class $= \frac{n(n-1)}{1 \,.\, 2}$

— — 3d Class $= \frac{n(n-1)\,(n-2)}{1 \,.\, 2 \,.\, 3}$

— — 4th Class $= \frac{n(n-1)\,(n-2)\,(n-3)}{1 \,.\, 2 \,.\, 3 \,.\, 4}$

. . . .

— — mth Class $= \frac{n(n-1)\,(n-2)\ldots\ldots(n-m+1)}{1 \,.\, 2 \,.\, 3 \ldots\ldots\ldots\ldots m}$

3) VARIATIONS.

a. VABIATIONS WITH REPETITIONS.

1) Var. (a, b, c, d, e, f)
(Second Class.)

aa	*ba*	*ca*	*da*	*ea*	*fa*
ab	*bb*	*cb*	*db*	*eb*	*fb*
ac	*bc*	*cc*	*dc*	*ec*	*fc*
ad	*bd*	*cd*	*dd*	*ed*	*fd*
ae	*be*	*ce*	*de*	*ee*	*fe*
af	*bf*	*cf*	*df*	*ef*	*ff*

2) Var. (a, b, c, d)
(Third Class.)

aaa
aab
aac
aad
aba
abb
abc
abd
aca
acb
acc
acd
ada
adb
adc
add
baa
bab
bac
bad
bba
bbb
bbc
bbd
bca
bcb
bcc
bcd
bda
bdb
bdc
bdd
caa
cab
cac
cad
cba
cbb
cbc
cbd
cca
ccb
ccc
ccd
cda
cdb
cdc
cdd
daa
dab
dac
dad
dba
dbb
dbc
dbd
dca
dcb
dcc
dcd
dda
ddb
ddc
ddd

3) Var. (*a b*, *c*)

(Fourth Class.)

aaaa
aaab
aaac
aaba
aabb
aabc
aaca
aacb
aacc
abaa
abab
abac
abba
abbb
abbc
abca
abcb
abcc
acaa
acab
acac
acba
acbb
acbc
acca
accb
accc
baaa
baab
baac
baba
babb
babc
baca
bacb
bacc
bbaa
bbab
bbac
bbba
bbbb
bbbc
bbca
bbcb
bbcc
bcaa
bcab
bcac
bcba
bcbb
bcbc
bcca
bccb
bccc
caaa
caab
caac
caba
cabb
cabc
caca
cacb
cacc
cbaa
cbab
cbac
cbba
cbbb
cbbc
cbca
cbcb
cbcc
ccaa
ccab
ccac
ccba
ccbb
ccbc
ccca
cccb
cccc

The number of variations with repetitions for n things and the mth Class is $= n^m$.

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b. VARIATIONS WITHOUT REPETITIONS.

Var. (a, b, c, d, e)
(Fourth Class.)

abcd
abce
abdc
abde
abec
abed
acbd
acbe
acdb
acde
aceb
aced
adbc
adbe
adcb
adce
adeb
adec
aebc
aebd
aecb
aecd
aedb
aedc
bacd
bace
badc
bade
baec
baed
bcad
bcae
bcda
bcde
bcea
bced
bdac
bdae
bdca
bdec
bdea
bdec
beac
bead
beca
becd
beda
bedc
cabd
cabe
cadb
cade
caeb
caed
cbad
cbae
cbda
cbde
cbea
cbed
cdab
cdae
cdba
cdbe
cdea
cdeb
ceab
cead
ceba
cebd
ceda
cedb
dabc
dabe
dacb
dace
daeb
daec
dbac
dbae
dbca
dbce
dbea
dbec
dcab
dcae
dcba
dcbe
dcea
dceb
deab
deac
deba
debc
deca
decb
eabc
eabd
eacb
eacd
eadb
eadc
ebac
ebad
ebca
ebcd
ebda
ebdc
ecab
ecad
ecba
ecbd
ecda
ecdb
edab
edac
edba
edbc
edca
edcb

The number of variations without repetitions for n things is

For the 1st Class $= n$
— — 2d Class $= n(n - 1)$
— — 3d Class $= n(n - 1)\ (n - 2)$
— — 4th Class $= n(n - 1)\ (n - 2)\ (n - 3)$
. . . .
— — mth Class $= n(n - 1)\ (n - 2) \ldots\ldots (n - m + 1)$

XI. THE BINOMIAL AND POLYNOMIAL THEOREMS FOR POSITIVE INTEGER EXPONENTS.

1) THE BINOMIAL THEOREM.

FORMULÆ.

I. $$(a \pm b)^n = a^n \pm \frac{n}{1} a^{n-1} b^1 + \frac{n(n-1)}{1\,.\,2} a^{n-2} b^2$$

$$\pm \frac{n(n-1)\,(n-2)}{1\,.\,2\;.\;3} a^{n-3} b^3 + \frac{n(n-1)\,(n-2)\,(n-3)}{1\,.\,2\;.\;3\;.\;4} a^{n-4} b^4$$

$$\pm \frac{n(n-1)\,(n-2)\,(n-3)\,(n-4)}{1\,.\,2\;.\;3\;.\;4\;.\;5} a^{n-5} b^5 + \ldots\ldots\ldots$$

$$\pm \frac{n(n-1)\,(n-2)\ldots\ldots\ldots\ldots\ldots\ldots 1}{1\,.\,2\;.\;3\;\ldots\ldots\ldots\ldots\ldots\ldots n} b^n$$

Of the two signs ±, which occur here, the upper $+$, belongs to the developement of $(a + b)^n$, and the lower $-$, to that of $(a - b)^n$. The last term is always $= b^n$, and is positive, when n is an even number, and negative when n is odd.

II. The number of terms of the series is $= n + 1$. The law of the exponents of a and b is apparent. The coefficients increase as far as the middle of the series, and then decrease in the same order and magnitude, so that the coefficients of the terms equally distant from the first and last terms are equal. When n is an even number, the developement of $(a + b)^n$, has a mean term, which is

$$+ \frac{n(n-1)\,(n-2)\,.\,.\,.\,.\,.\,.\,.\left(\frac{n}{2}+1\right)}{1\,.\,2\;.\;3\quad.\,.\,.\,.\,.\,.\,.\frac{n}{2}} a^{\frac{n}{2}} b^{\frac{n}{2}}$$

In the developement of $(a - b)^n$, their middle term is negative, when n can be represented under the form $4m+2$; otherwise it is positive. When n is an odd number, there are two middle terms, and these are in the developement of $(a+b)^n$,

$$+\frac{n(n-1)\ldots.\frac{n+3}{2}}{1\,.\,2\ \ldots.\frac{n-1}{2}}a^{\frac{n+1}{2}}b^{\frac{n-1}{2}}+\frac{n(n-1)\ldots.\frac{n+1}{2}}{1\,.\,2\ \ldots.\frac{n+1}{2}}a^{\frac{n-1}{2}}b^{\frac{n+1}{2}}$$

$$=\frac{n(n-1)\ldots.\frac{n+3}{2}}{1\,.\,2\ \ldots.\frac{n-1}{2}}\left(+a^{\frac{n+1}{2}}b^{\frac{n-1}{2}}+a^{\frac{n-1}{2}}b^{\frac{n+1}{2}}\right).$$

In the developement of $(a - b)^n$, the first of the above terms is positive, and the second negative, when the number n is such that it can be represented under the form $4m+1$; the reverse order of signs takes place when the number n can be represented by $4m+3$.

III. The general $(m+1)$th term of the series is

$$\pm\frac{n(n-1)}{1\,.\,2}\,\frac{(n-2)\ldots\ldots(n-m+1)}{.\quad 3\ldots\ldots\ldots\ldots\ldots\ldots\ldots m}\,a^{n-m}b^m$$

of the two signs $\pm$, the upper one $+$, belongs to all the terms of the developement $(a+b)^n$, but the lower one $-$, only to the even terms of $(a+b)^n$.

IV. The sum of the coefficients in the developement of

$$(a+b)^n,\ \text{that is, } 1+\frac{n}{1}+\frac{n(n-1)}{1\,.\,2}+,\ \&c,\ \text{is} =$$
$$(1+1)^n=2^n;$$

and the sum of all coefficients in the developement of

$(a-b)^n$, that is, the sum of $1-\frac{n}{1}+\frac{n(n-1)}{1\,.\,2}-$ &c, is $=(1-1)^n=0$.

V. If $\frac{b}{a}$ be put $= Q$, and the first term of the developement be represented by A, the second by B, the third by C, &c, then we shall have

$$(a\pm b)^n = A \pm \frac{n}{1}AQ + \frac{n-1}{2}BQ \pm \frac{n-2}{2}CQ + \frac{n-3}{4}DQ$$
$$\pm \frac{n-4}{5}EQ \ldots\ldots;$$

which formula is very convenient in practice, in order to obtain the following terms from the preceding ones.

EXAMPLES.

1) $(a \pm b)^1 = a \pm b$

2) $(a \pm b)^2 = a^2 \pm 2ab + b^2$

3) $(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$

4) $(a \pm b)^4 = a^4 \pm 4a^3b + 6a^2b^2 \pm 4ab^3 + b^4$

5) $(a \pm b)^5 = a^5 \pm 5a^4b + 10a^3b^2 \pm 10a^2b^3 + 5ab^4 \pm b^5$

6) $(a \pm b)^6 = a^6 \pm 6a^5b + 15a^4b^2 \pm 20a^3b^3 + 15a^2b^4 \pm 6ab^5 + b^6$

7) $(a \pm b)^7 = a^7 \pm 7a^6b + 21a^5b^2 \pm 35a^4b^3 + 35a^3b^4 \pm 21a^2b^5 + 7ab^6 \pm b^7$

8) $(a \pm b)^8 = a^8 \pm 8a^7b + 28a^6b^2 \pm 56a^5b^3 + 70a^4b^4 \pm 56a^3b^5 + 28a^2b^6 \pm 8ab^7 + b^8$

9) $(a \pm b)^9 = a^9 \pm 9a^8b + 36a^7b^2 \pm 84a^6b^3 + 126a^5b^4 \pm 126a^4b^5 + 84a^3b^6 \pm 36a^2b^7 + 9ab^8 \pm b^9$

10) $(a \pm b)^{10} = a^{10} \pm 10a^9b + 45a^8b^2 \pm 120a^7b^3 + 210a^6b^4 \pm 252a^5b^5 + 210a^4b^6 \pm 120a^3b^7 + 45a^2b^8 \pm 10ab^9 + b^{10}$

11) $(1 \pm x)^{11} = 1 \pm 11x + 55x^2 \pm 165x^3 + 330x^4 \pm 462x^5 + 462x^6 \pm 330x7 + 165x^8 \pm 52x^9 + 11x^{10} \pm x^{11}$

12) $(1 \pm x)^{12} = 1 \pm 12x + 66x^2 \pm 220x^3 + 495x^4 \pm 792x^5 + 924x^6 \pm 792x^7 + 495x^8 \pm 220x^9 + 66x^{10} \pm 12x^{11} + x^{12}$

13) $(5-4x)^4 = 625 - 2000x + 2400x^2 - 1280x^3 + 256x^4$

14) $(3 - 2x^2)^6 = 729 - 2916x^2 + 4860x^4 - 4320x^6 + 2160x^8 - 576x^{10} + 64x^{12}$

15) $(\frac{1}{2}x + 2y)^7 = \frac{1}{128}x^7 + \frac{7}{32}x^6y + \frac{21}{8}x^5y^2 + \frac{35}{2}x^4y^3 + 70x^3y^4 + 168x^2y^5 + 224xy^6 + 128y^7$

16) $(a^3 + 3ab)^9 = a^{27} + 27a^{25}b + 324a^{23}b^2 + 2268a^{21}b^3 + 10206a^{19}b^4 + 30618a^{17}b^5 + 61236a^{15}b^6 + 78732a^{13}b^7 + 59049a^{11}b^8 + 19683a^9b^9$

17) $(3ac - 2bd)^5 = 243a^5c^5 - 810a^4c^4bd + 1080a^3c^3b^2d^2 - 720a^2c^2b^3d^3 + 240acb^4d^4 - 32b^5d^5$

18) $(5a^2c^2d - 4abd^2)^4 = 625a^8c^8d^4 - 2000a^7bc^6d^5 + 2400a^6b^2c^4d^6 - 1280a^5b^3c^2d^7 + 256a^4b^4d^8$

19) $\left(\frac{2ac}{b^2} + \frac{1}{4}bc^2d\right)^6 = 64a^6c^6b^{-12} + 48a^5c^7db^{-9} + 15a^4c^8d^2b^{-6} + \frac{5}{2}a^3c^9d^3b^{-3} + \frac{15}{64}a^2c^{10}d^4 + \frac{3}{356}ab^3c^{11}d^5 + \frac{1}{4096}b^6c^{12}d^6$

20) $(\sqrt{a} \pm \sqrt{b})^4 = a^2 + 6ab + b^2 \pm (4a + 4b)\sqrt{ab}$

21) $(\sqrt{a} \pm \sqrt{b})^7 = (a^3 + 21a^2b + 35ab^2 + 7b^3)\sqrt{a} \pm (7a^3 + 35a^2b + 21ab^2 + b^3)\sqrt{b}$

22) $$(a+b)^n+(a-b)^n=2\left(a^n+\frac{n(n-1)}{1\,.\,2}a^{n-}b^2+\frac{n(n-1)\,(n-2)\,(n-3)}{1\,.\,2\;.\;3\;.\;4}a^{n-4}b^4+\frac{n....n-5}{1......6}a^{n-6}b^6+\ldots\ldots\right)$$[*]

23) $$(a+b)^n-(a-b)^n=2\left(\frac{n}{1}a^{n-1}b+\frac{n(n-1)\,(n-2)}{1\,.\,2\;.\;3}a^{n-3}b^3+\frac{n...n-4}{1.......5}a^{n-5}b^5+\frac{n...n-6}{1......7}a^{n-7}b^7+\ldots\ldots\right)$$

24) $$(a\pm b\sqrt{-1})^n=a^n-\frac{n(n-1)}{1\,.\,2}a^{n-2}b^2+\frac{n....n-3}{1......4}a^{n-4}b^4-\&c\pm\left(\frac{n}{1}a^{n-1}b-\frac{n(n-1)\,(n-2)}{1\,.\,2\;.\;3}a^{n-3}b^3+\frac{n....n-4}{1........5}a^{n-5}b^5-\&c.\right)\sqrt{-1}$$

25) $$(a+b\sqrt{-1})^n+(a-b\sqrt{-1})^n=2\left[a^n-\frac{n(n-1)}{1\,.\,2}a^{n-2}b^2+\frac{n....n-3}{1........4}a^{n-4}b^4-\&c.\right]$$[†]

26) $$\frac{(a+b\sqrt{-1})^n-(a-b\sqrt{-1})^n}{\sqrt{-1}}=2\left[na^{n-1}b-\frac{n.n-1.n-2}{1\,.\,2\;.\;3}a^{n-3}b^3+\frac{n.....n-4}{1.........5}a^{n-5}b^5-\&c.\right]$$

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[*] The developements 22, 33, 24, 25, 26, are continued till they terminate, that is, until the coefficients become 0.

[†] Consequently, $(a+b\sqrt{-1})^n+(a-b\sqrt{-1})^n$ and $[(a+b\sqrt{-1})^n-(a-b\sqrt{-1})^n]:\sqrt{-1}$, are *real* quantities.

27) The 3d term of $(a+b)^{15}$ is $= 105a^{13}b^2$

28) — 5th — — $(a+b)^{16}$ is $= 1820a^{12}b^4$

29) — 6th — — $(a-b)^{30}$ is $= -142506a^{25}b^5$

30) — 4th — — $(a-b)^{100}$ is $= -161700a^{97}b^5$

31) — 5th — — $(a^2-b^2)^{12}$ is $= 495a^{16}b^8$

32) — 9th — — $(2ab-cd)^{14}$ is $= 192192a^6b^6c^8d^8$

33) The middle term of $(a-b)^{16}$ is $= 12870a^8b^8$

34) — — — — $(a-b)^{18}$ is $= -48620a^9b^9$

35) The two middle terms of $(a-b)^{17}$ are

$24310a^9b^8 - 24310a^8b^9$

36) — — — — — $(a-b)^{19}$ are

$-92378a^{10}b^9 + 92378a^9b^{10}$

2) THE POLYNOMIAL THEOREM.

FORMULÆ.

I. The nth power of the Polynomial $a+b+c+d+\ldots$ $\ldots+x$ consists of the sum of all the combinations with repetitions of the quantities $a, b, c, d \ldots\ldots x$ for the nth Class; each of these combinations taken as many times, as the elements of which it consists, (of these one or more are always equal) admit of permutations.

II. If, therefore, the polynomial $a + b + c + d + \dots x$ consists of m terms, then the nth power contains

$$\frac{m(m+1)\,(m+2)\,.\,.\,.\,.\,.\,.\,.\,(m+n-1)}{1\;.\;2\;.\;3\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,n}$$

terms. The sum of the exponents of a, b, c, d, ... x, in each term is $= n$, and the sum of all the coefficients $= m^n$.

III. Putting, for the sake of brevity, $b + c + d + e + \dots \dots x = p$, we have

$$(a + b + c + d + \dots\dots + x)^n = (a + p)^n =$$

$$a^n + \frac{n}{1}a^{n-1}p + \frac{n(n-1)}{1\;.\;2}a^{n-2}p^2 + \dots\dots + p^n;$$

which formula may be applied with advantage in many cases.

EXAMPLES.

1) $(a + b + c)^2 = a^2 + 2ab + 2ac + b^2 + 2bc + c^2$

2) $(a + b + c)^3 = a^3 + 3a^2b + 3a^2c + 3ab^2 + 6abc$
$+ 3ac^2 + b^3 + 3b^2c + 3bc^2 + c^3$

3) $(a + b + c)^4 = a^4 + 4a^3b + 4a^3c + 6a^2b^2 + 12a^2bc$
$+ 6a^2c^2 + 4ab^3 + 12ab^2c + 12abc^2 + 4ac^3 + b^4$
$+ 4b^3c + 6b^2c^2 + 4bc^3 + c^4$

4) $(a + b + c)^5 = a^5 + 5a^4b + 5a^4c + 10a^3b^2 + 20a^3bc$
$+ 10a^3c^2 + 10a^2b^3 + 30a^2b^2c + 30a^2bc^2 + 10a^2c^3$
$+ 5ab^4 + 20ab^3c + 30ab^2c^2 + 20abc^3 + 5ac^4 + b^5$
$+ 5b^4c + 10b^3c^2 + 10b^2c^3 + 5bc^4 + c^5$

5) $(a + b + c)^6 = a^6 + 6a^5b + 6a^5c + 15a^4b^2 + 30a^4bc$
$+ 15a^4c^2 + 20a^3b^3 + 60a^3b^2c + 60a^3bc^2 + 20a^3c^3$
$+ 15a^2b^4 + 60a^2b^3c + 90a^2b^2c^2 + 60a^2bc^3 + 15a^2c^4$
$+ 6ab^5 + 30ab^4c + 60ab^3c^2 + 60ab^2c^3 + 30abc^4$
$+ 6ac^5 + b^6 + 6b^5c + 15b^4c^2 + 20b^3c^3 + 15b^2c^4$
$+ 6bc^5 + c^6$

6) $(a + b + c)^7 = a^7 + 7a^6b + 7a^6c + 21a^5b^2 + 42a^5bc + 21a^5c^2 + 35a^4b^3 + 105a^4b^2c + 105a^4bc^2 + 35a^4c^3 + 35a^3b^4 + 140a^3b^3c + 210a^3b^2c^2 + 140a^3bc^3 + 35a^3c^4 + 21a^2b^5 + 105a^2b^4c + 210a^2b^3c^2 + 210a^2b^2c^3 + 105a^2bc^4 + 21a^2c^5 + 7ab^6 + 42ab^5c + 105ab^4c^2 + 140ab^3c^3 + 105ab^2c^4 + 42abc^5 + 7ac^6 + b^7 + 7b^6c + 21b^5c^2 + 35b^4c^3 + 35b^3c^4 + 21b^2c^5 + 7bc^6 + c^7$

7) $(a + b + c + d)^2 = a^2 + 2ab + 2ac + 2ad + b^2 + 2bc + 2bd + c^2 + 2cd + d^2$

8) $(a + b + c + d)^3 = a^3 + 3a^2b + 3a^2c + 3a^2d + 3ab^2 + 6abc + 6abd + 3ac^2 + 6acd + 3ad^2 + b^3 + 3b^2c + 3b^2d + 3bc^2 + 6bcd + 3bd^2 + c^3 + 3c^2d + 3cd^2 + d^3$

9) $(a + b + c + d)^4 = a^4 + 4a^3b + 4a^3c + 4a^3d + 6a^2b^2 + 12a^2bc + 12a^2bd + 6a^2c^2 + 12a^2cd + 6a^2d^2 + 4ab^3 + 12ab^2c + 12ab^2d + 12abc^2 + 24abcd + 12abd^2 + 4ac^3 + 12ac^2d + 12acd^2 + 4ad^3 + b^4 + 4b^3c + 4b^3d + 6b^2c^2 + 12b^2cd + 6b^2d^2 + 4bc^3 + 12bc^2d + 12bcd^2 + 4bd^3 + c^4 + 4c^3d + 6c^2d^2 + 4cd^3 + d^4$

10) $(a + b + c + d)^5 = a^5 + 5a^4b + 5a^4c + 5a^4d + 10a^3b^2 + 20a^3bc + 20a^3bd + 10a^3c^2 + 20a^3cd + 10a^3d^2 + 10a^2b^3 + 30a^2b^2c + 30a^2b^2d + 30a^2bc^2 + 60a^2bcd + 30a^2bd^2 + 10a^2c^3 + 30a^2c^2d + 30a^2cd^2 + 10a^2d^3 + 5ab^4 + 20ab^3c + 20ab^3d + 30ab^2c^2 + 60ab^2cd + 30ab^2d^2 + 20abc^3 + 60abc^2d + 60\,abcd^2 + 20abd^3 + 5ac^4 + 20ac^3d + 30ac^2d^2 + 20acd^3 + 5ad^4 + b^5 + 5b^4c + 5b^4d + 10b^3c^2 + 20b^3cd + 10b^3d^2 + 10b^2c^3 + 30b^2c^2d + 30b^2cd^2 + 10b^2d^3 + 5bc^4 + 20\,bc^3d + 30bc^2d^2 + 20bcd^3 + 5bd^4 + c^5 + 5c^4d + 10c^3d^2 + 10c^2d^3 + 5cd^4 + d^5.$

11) $(a+2b-c)^3 = a^3 + 6a^2b - 3a^2c + 12ab^2 - 12abc$
$+ 3ac^2 + 8b^3 - 12b^2c + 6bc^2 - c^3$

12) $\left(3a-5b-\frac{2c}{3}\right)^4 = 81a^4 - 540a^3b - 72a^3c + 1350a^2b^2$
$+ 360a^2bc + 24a^2c^2 - 1500ab^3 - 600ab^2c -$
$80abc^2 - \frac{32}{9}ac^3 + 625b^4 + \frac{1000}{3}b^3c + \frac{200}{3}b^2c^2$
$+ \frac{160}{27}bc^3 + \frac{16}{81}c^4$

13) $(7a^2 + 3ab + 4b^2)^3 = 343a^6 - 441a^5b + 777a^4b^2$
$- 531a^3b^3 + 444a^2b^4 + 144ab^5 + 64b^6$

14) $\left(\frac{2b^3c}{3a^4} + 7a^2b - \frac{1}{2}\right)^5 = \frac{32}{243}a^{-20}b^{15}c^5 + \frac{560}{81}a^{-14}b^{13}c^4$
$- \frac{40}{81}a^{-16}b^{12}c^4 + \frac{3920}{27}a^{-8}b^{11}c^3 - \frac{560}{27}a^{-10}b^{10}c^3 +$
$\frac{20}{27}a^{-12}b^9c^3 + \frac{13720}{9}a^{-2}b^9c^2 - \frac{980}{3}a^{-4}b^8c^2 +$
$\frac{70}{3}a^{-6}b^7c^2 - \frac{5}{9}a^{-8}b^6c^2 - \frac{24010}{3}a^4b^7c - \frac{6860}{3}a^2b^6c$
$+ 245b^5c - \frac{35}{3}a^{-2}b^4c + \frac{5}{24}a^{-4}b^3c + 16807a^{10}b^5$
$- \frac{12005}{2}a^8b^4 + \frac{1715}{2}a^6b^3 - \frac{245}{4}a^4b^2 + \frac{35}{16}a^2b - \frac{1}{32}$

15) $\left(\frac{2ab}{c^m} - 5a^3c^{3m} - 3ab^2 + \frac{b}{2a}\right)^3 = 8a^3b^3c^{-3m} -$
$60a^5b^2c^m - 36a^3b^4c^{-2m} + 6ab^3c^{-2m} + 150a^7bc^{5m}$
$+ 180a^5b^3c^{2m} - 30a^3b^2c^{2m} + 54a^3b^5c^{-m} - 18ab^4c^{-m}$
$+ \frac{3}{2}a^{-1}b^3c^{-m} - 125a^9c^{9m} - 225a^7b^2c^{6m} + \frac{75}{2}a^5bc^{6m}$
$+ 135a^5b^4c^{3m} + 45a^3b^3c^{3m} - \frac{15}{4}ab^2c^{3m} - 27a^3b^6$
$+ \frac{27}{2}ab^5 - \frac{9}{4}a^{-1}b^4 + \frac{1}{8}a^{-3}b^3$

XII. PROGRESSIONS.

1) ARITHMETIC PROGRESSIONS.

I. Let a denote the first term, n, the number of terms, d, the equidifference, and s, the sum of an arithmetical progression, then we have

1) $t = a + (n - 1)d$

2) $s = (a + t)\frac{n}{2} = [2a + (n - 1)d]\frac{n}{2}$.

By means of these two formulæ the values of t and s may be found, when a, d, and n are given.

EXAMPLES.

N	Given Values.			Required Values.	
1	$a=1$,	$d=1$,	$n=14$	$t=14$,	$s=105$
2	$a=2$,	$d=3$,	$n=17$	$t=50$,	$s=442$
3	$a=7$,	$d=\frac{1}{4}$,	$n=16$	$t=10\frac{3}{4}$,	$s=142$
4	$a=2\frac{1}{2}$,	$d=\frac{1}{3}$,	$n=100$	$t=35\frac{1}{2}$,	$s=1900$
5	$a=\frac{3}{4}$,	$d=\frac{1}{8}$	$n=26$	$t=3\frac{7}{8}$,	$s=60\frac{1}{8}$
6	$a=\frac{5}{7}$,	$d=1\frac{2}{3}$	$n=13$	$t=20\frac{5}{7}$,	$s=139\frac{2}{7}$
7	$a=-7$,	$d=3$,	$n=8$	$t=14$,	$s=28$
8	$a=-6$,	$d=\frac{3}{4}$,	$n=30$	$t=15\frac{3}{4}$,	$s=146\frac{1}{4}$
9	$a=\frac{1}{2}$,	$d=-\frac{1}{8}$,	$n=20$	$t=-1\frac{7}{8}$,	$s=-13\frac{3}{4}$
10	$a=3\frac{1}{3}$,	$d=-2\frac{5}{6}$,	$n=15$	$t=-36\frac{1}{3}$,	$s=-247\frac{1}{2}$
11	$a=0$,	$d=\frac{1}{2}$,	$n=11$	$t=5$,	$s=27\frac{1}{2}$
12	$a=-10$,	$d=-2$,	$n=6$	$t=-20$,	$s=-90$
13	$a=-\frac{3}{4}$	$d=-\frac{7}{8}$,	$n=25$	$t=-21\frac{3}{4}$,	$s=-281\frac{1}{4}$

II. When three of the five magnitudes ***a, d, n, t, s,*** **are** given, the other two may be found by the following table;

TABLE OF FORMULÆ FOR ARITHMETICAL PROGRESSION.

N	Given.	Required.	Formulæ.
1	a, d, n	t	$t = a + (n-1)d$
2	a, d, s		$t = -\frac{1}{2}d \pm \sqrt{[2ds + (a - \frac{1}{2}d)^2]}$
3	a, n, s		$t = \frac{2s}{n} - a$
4	d, n, s		$t = \frac{s}{n} + \frac{(n-1)d}{2}$
5	a, d, n	s	$s = \frac{1}{2}n[2a + (n-1)d]$
6	a, d, t		$s = \frac{a+t}{2} + \frac{(t+a)(t-a)}{2d}$
7	a, n, t		$s = \frac{1}{2}n(a+t)$
8	d, n, t		$s = \frac{1}{2}n[2t - (n-1)d]$
9	a, n, t	d	$d = \frac{t-a}{n-1}$
10	a, n, s		$d = \frac{2s - 2an}{n(n-1)}$
11	a, t, s		$d = \frac{(t+a)(t-a)}{2s - t - a}$
12	n, t, s		$d = \frac{2nt - 2s}{n(n-1)}$

N	Given.	Required.	Formulæ.
13	a, d, t	n	$n = 1 + \frac{t - a}{d}$
14	a, d, s		$n = \frac{d - 2a}{2d} \pm \sqrt{\left[\frac{2s}{d} + \left(\frac{2a - d}{2d}\right)^2\right]}$
15	a, t, s		$n = \frac{2s}{a + t}$
16	d, t, s		$n = \frac{2t + d}{2d} \pm \sqrt{\left[\left(\frac{2t + d}{2d}\right)^2 - \frac{2s}{d}\right]}$
17	d, n, t	a	$a = t - (n - 1)d$
18	d, n, s		$a = \frac{s}{n} - \frac{(n - 1)d}{2}$
19	d, t, s		$a = \frac{1}{2}d \pm \sqrt{[(t + \frac{1}{2}d)^2 - 2ds]}$
20	n, t, s		$a = \frac{2s}{n} - t$

* * *

What is meant by an arithmetic progression of the 1st, 2d, 3d, &c, order? and how are the successive orders derived from the series of the 1st order a, $a + d$, $a + 2d$, $a + 3d$, &c?

What are figurate numbers, and how are they derived from the series 1, $1 + d$, $1 + 2d$, &c? What, particularly, are polygonal and pyramidal numbers?*

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* The correctness of the general terms in the following series may be easily proved, merely by subtraction, or conversely by addition. Thus, we need but subtract from the general term of

SERIES OF FIGURATES OF THE 1ST ORDER, BEGINNING WITH THE TERM 1.

$1, 2, 3, \ 4, \ 5, \ 6 \ldots\ldots\ldots \ n$

$1, 3, 5, \ 7, \ 9, 11 \ldots\ldots\ldots 2n - 1$

$1, 4, 7, 10, 13, 16 \ldots\ldots\ldots 3n - 2$

$1, 5, 9, 13, 17, 21 \ldots\ldots\ldots 4n - 3$

. . . .

$1, 1 + d, 1 + 2d, 1 + 3d \ldots\ldots dn - d + 1$

POLYGONAL NUMBERS.

$1, 3, \ 6, 10, 15, 21 \ldots\ldots\ldots \dfrac{n(n+1)}{1 \,.\, 2}$

$1, 4, \ 9, 16, 25, 36 \ldots\ldots\ldots n^2$

$1, 5, 12, 22, 35, 51 \ldots\ldots\ldots \dfrac{n(3n-1)}{1 \,.\, 2}$

$1, 6, 15, 28, 45, 66 \ldots\ldots\ldots n(2n-1)$

. . . .

$1, 2 + d, 3 + 3d, 4 + 6d \ldots\ldots \dfrac{n(dn - d + 2)}{1 \,.\, 2}$

---

each series that immediately preceding, and we shall obtain the general term of the series from which it is derived. For example, from the general term of the triangular numbers $\frac{n.(n+1)}{1 \,.\, 2}$, we obtain the general term of the natural numbers

$$\frac{n.(n+1)}{1 \,.\, 2} - \frac{n.(n-1)}{1 \,.\, 2} = n.$$

PYRAMIDAL NUMBERS.

1, 4, 10, 20, 35, 56 .... $\frac{n(n+1)(n+2)}{1\,.\,2\,.\,3}$

1, 5, 14, 30, 55, 91 .... $\frac{n(n+1)(2n+1)}{1\,.\,2\,.\,3}$

1, 6, 18, 40, 75, 126 .... $\frac{n^2(n+1)}{1\,.\,2}$

1, 7, 22, 50, 95, 161 .... $\frac{n(n+1)(4n-1)}{1\,.\,2\,.\,3}$

. . . .

1, $3+d$, $6+4d$, $10+10d$ ..... $\frac{n(n+1)(dn-d+3)}{1\,.\,2\,.\,3}$

SERIES WHICH ARISE FROM THE CONTINUAL ADDITION OF THE NATURAL NUMBERS.

1, 2, 3, 4, 5, 6 ...... n

1, 3, 6, 10, 15, 21 ...... $\frac{n(n+1)}{1\,.\,2}$

1, 4, 10, 20, 35, 56 ...... $\frac{n(n+1)(n+2)}{1\,.\,2\,.\,3}$

1, 5, 15, 35, 70, 126 ...... $\frac{n(n+1)(n+2)(n+3)}{1\,.\,2\,.\,3\,.\,4}$

and so on.

2) GEOMETRICAL PROGRESSIONS.

I. If a is the first term, e the common ratio, t the last term, n the number of terms, and s the sum of them; then

$$1)\quad t = ae^{n-1}$$

$$2)\quad s = \frac{et - a}{e - 1} = \frac{a(e^n - 1)}{e - 1}$$

By means of these two formulæ, the values of t and s may be found, when a, e, n, are given.

N	Given Values			Required Values.	
1	$a=1$,	$e=2$,	$n=7$	$t=64$,	$s=127$
2	$a=4$,	$e=3$,	$n=10$	$t=78732$,	$s=118096$
3	$a=5$,	$e=4$,	$n=9$	$t=327680$,	$s=436905$
4	$a=9$,	$e=\frac{7}{4}$,	$n=7$	$t=258\frac{2073}{4096}$,	$s=591\frac{741}{4096}$
5	$a=6\frac{1}{4}$,	$e=\frac{3}{2}$,	$n=8$	$t=106\frac{403}{512}$,	$s=307\frac{441}{512}$
6	$a=6$,	$e=\frac{3}{4}$,	$n=6$	$t=1\frac{217}{512}$,	$s=19\frac{373}{512}$
7	$a=8$,	$e=\frac{1}{2}$,	$n=15$	$t=\frac{1}{2048}$,	$s=15\frac{2047}{2048}$
8	$a=3\frac{1}{2}$,	$e=\frac{3}{5}$,	$n=8$	$t=\frac{15309}{131250}$,	$s=8\frac{45237}{75625}$
9	$a=\frac{5}{6}$,	$e=\frac{2}{3}$,	$n=11$	$t=\frac{2560}{177147}$,	$s=2\frac{166907}{354294}$
10	$a=3$,	$e=\frac{7}{5}$,	$n=25$	$t=9642,59...$,	$s=33741,59...$
11	$a=7\frac{1}{2}$,	$e=\frac{27}{20}$,	$n=31$	$t=60964,11...$,	$s=235125,85...$
12	$a=63$,	$e=\frac{157}{132}$,	$n=58$	$t=1238530,19...$,	$s=7777637,01...$
13	$a=5560$,	$e=\frac{9}{11}$,	$n=40$	$t=2,219309...$	$s=30570,01310...$
14	$a=393\frac{1}{3}$,	$e=\frac{5}{12}$,	$n=17$	$t=0,0003246241...$,	$s=674,2854824...$
15	$a=1$,	$e=\frac{1}{2}$,	$n=\infty$	$t=0$,	$s=2$
16	$a=40$,	$e=\frac{3}{7}$,	$n=\infty$	$t=0$,	$s=70$
17	$a=9$,	$e=\frac{2}{3}$,	$n=\infty$	$t=0$,	$s=27$

In the examples, 10, 11, 12, 13, 14, the values of t are best found by logarithms, from which those of s are then easily found.

Ques. What is the sum of n terms of the geometrical progression

$$a,\, b,\, \frac{b^2}{a},\, \frac{b^3}{a^2},\, \frac{b^4}{a^3},\, \ldots\ldots \frac{b^{n-1}}{a^{n-2}}?$$

And what is the sum of the same series, when the number of terms becomes infinite?

Ans. The sum of the finite progression is

$$= \frac{b^n - a^n}{(b - a)\,a^{n-2}} = \frac{a^n - b^n}{(a - b)\,a^{n-2}};$$

the sum of the infinite one is $= \dfrac{a^2}{a - b}$.

What is the sum of the infinite geometrical progression

$$a - b + \frac{b^2}{a} - \frac{b^3}{a^2} + \frac{b^4}{a^3} - \&c?$$

Ans. $\dfrac{a^2}{a + b}$.

What vulgar fraction is equivalent to the recurring decimal fraction

$0{,}868686\ldots\ldots \doteq 86(0{,}01 + 0{,}0001 + 0{,}000001 + \ldots\ldots)$?

Ans. $\frac{86}{99}$.

What vulgar fraction is equal to the decimal 0,375375....?

Ans. $\frac{375}{999} = \frac{125}{333}$.

And what, to the decimal fraction 0,142857, ... whose period is 142857?

Ans. $\frac{142857}{999999} = \frac{1}{7}$.

Thus every recurring decimal fraction may be transformed into a vulgar fraction.

II. When any three of the magnitudes a, e, n, t, s, are given, the other two may be found, as in the following table.

FORMULÆ FOR GEOMETRICAL PROGRESSIONS.

N	Given.	Required.	Formulæ.
1	a, e, n	t	$t = ae^{n-1}$
2	a, e, s		$t = \frac{a + (e-1)s}{e}$
3	a, n, s		$t(s-t)^{n-1} - a(s-a)^{n-1} = 0$
4	e, n, s		$t = \frac{(e-1)se^{n-1}}{e^n - 1}$
5	a, e, n	s	$s = \frac{a(e^n - 1)}{e - 1}$
6	a, e, t		$s = \frac{et - a}{e - 1}$
7	a, n, t		$s = \frac{t^{\frac{n}{n-1}} - a^{\frac{n}{n-1}}}{t^{\frac{1}{n-1}} - a^{\frac{1}{n-1}}}$
8	e, n, t		$s = \frac{t(e^n - 1)}{(e-1)e^{n-1}}$
9	e, n, t	a	$a = \frac{t}{e^{n-1}}$
10	e, n, s		$a = \frac{(e-1)s}{e^n - 1}$
11	e, t, s		$a = et - (e-1)s$
12	n, t, s		$a(s-a)^{n-1} - t(s-t)^{n-1} = 0$

N	Given.	Required.	Formulæ.
13	$a,\ n,\ t$	e	$e=\sqrt[n-1]{\frac{t}{a}}$
14	$a,\ n,\ s$		$e^n-\frac{s}{a}e+\frac{s-a}{a}=0$
15	$a,\ t,\ s$		$e=\frac{s-a}{s-t}$
16	$n,\ t,\ s$		$e^n-\frac{s}{s-t}e^{n-1}+\frac{t}{s-t}=0$
17	$a,\ e,\ t$	n	$n=\frac{\log t-\log a}{\log e}+1$
18	$a,\ e,\ s$		$n=\frac{\log\,[a+(e-1)s]-\log a}{\log e}$
19	$a,\ t,\ s$		$n=\frac{\log t-\log a}{\log\,(s-a)-\log\,(s-t)}+1$
20	$e,\ t,\ s$		$n=\frac{\log t-\log\,[et-(e-1)s]}{\log e}+1$

XI. CONTINUED FRACTIONS.

1) CONTINUED FRACTIONS IN GENERAL.

I. A continued fraction is one of the following form.

$$\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e+\&c.}}}}}$$

the meaning of which is perceptible from the manner in which it is written. It is also assumed, that the quantities a, b, c, d, e, &c, which are commonly called *quotients*, are all positive integers. The fractions

$$\frac{1}{a},\ \frac{1}{a+\frac{1}{b}},\ \frac{1}{a+\frac{1}{b+\frac{1}{c}}},\ \&c,$$

are called *Approximating Fractions*, because they approach the more nearly the given fraction, the farther they are continued.

II. If these approximating fractions be transformed into vulgar fractions, we obtain the following approximations to the given fractions.

1) $\frac{1}{a}$

2) $\frac{b}{ab+1}$

3) $\frac{bc+1}{(ab+1)\,c+a}$

4) $\frac{(bc+1)\,d+b}{(abc+c+a)\,d+ab+1}$

5) $\frac{(bcd+d+b)\,e+bc+1}{(abcd+cd+ad+ab+1)\,e+abc+c+a}$

III. These approximating values may be derived from one another. Let $\frac{R}{S}$ be the $(n-1)$th of them, and $\frac{T}{V}$ the $(n-2)$d; farther, let q be the nth of the quotients, a, b, c, d, e, &c; then the nth approximating value is

$$=\frac{qR+T}{qS+V}.$$

EXAMPLES.

N	Given Fractions.	Quotients.	Approximating Values.
1	$\frac{351}{965}$	2, 1, 2, 1, 87	$\frac{1}{2}$, $\frac{1}{3}$, $\frac{3}{8}$, $\frac{4}{11}$
2	$\frac{251}{764}$	3, 22, 1, 4, 2	$\frac{1}{3}$, $\frac{22}{67}$, $\frac{23}{70}$, $\frac{114}{347}$
3	$\frac{1769}{5537}$	3, 7, 1, 2, 4, 5, 1, 2	$\frac{1}{3}$, $\frac{7}{22}$, $\frac{8}{25}$, $\frac{23}{72}$, $\frac{100}{313}$, $\frac{523}{1637}$, $\frac{623}{1950}$
4	$\frac{907}{18564}$	20, 2, 7, 5, 2, 1, 3	$\frac{1}{20}$, $\frac{2}{41}$, $\frac{15}{307}$, $\frac{77}{1576}$, $\frac{169}{3459}$, $\frac{246}{5035}$
5	$\frac{1947}{3359}$	1, 1, 2, 1, 1, 1, 3, 2, 1, 1, 2, 3	$\frac{1}{1}$, $\frac{1}{2}$ $\frac{3}{5}$, $\frac{4}{7}$, $\frac{7}{12}$, $\frac{11}{19}$, $\frac{40}{69}$, $\frac{91}{157}$, $\frac{131}{226}$, $\frac{222}{383}$, $\frac{575}{992}$
6	$\frac{587}{1943}$	3, 3, 4, 2, 3, 1, 1, 2	$\frac{1}{3}$, $\frac{3}{10}$, $\frac{13}{43}$, $\frac{29}{96}$, $\frac{100}{331}$, $\frac{129}{427}$, $\frac{229}{758}$
7	$\frac{5065}{13891}$	2, 1, 2, 1, 7, 1, 1, 1, 2, 1, 13	$\frac{1}{2}$, $\frac{1}{3}$, $\frac{5}{8}$, $\frac{4}{11}$, $\frac{31}{85}$, $\frac{35}{96}$, $\frac{66}{181}$, $\frac{101}{277}$, $\frac{268}{735}$, $\frac{369}{1012}$
8	$\frac{5743}{80937}$	14, 10, 1, 2, 1, 3, 3, 3, 3	$\frac{1}{14}$, $\frac{10}{141}$, $\frac{11}{155}$, $\frac{32}{451}$, $\frac{43}{606}$, $\frac{161}{2269}$, $\frac{526}{7413}$, $\frac{1739}{24508}$
9	$\frac{13957}{59476}$	4, 3, 1, 4, 1, 2, 1, 11, 2, 6	$\frac{1}{4}$, $\frac{3}{13}$, $\frac{4}{17}$, $\frac{19}{81}$, $\frac{23}{98}$, $\frac{65}{277}$, $\frac{88}{375}$ $\frac{1033}{4402}$, $\frac{2154}{9179}$
10	$\frac{3215763}{94218374}$	29, 3, 2, 1, 8, 1, 1, 6, &c.	$\frac{1}{29}$, $\frac{3}{88}$, $\frac{7}{205}$, $\frac{10}{293}$, $\frac{87}{2549}$, $\frac{97}{2842}$, $\frac{184}{5391}$, &c.

The lunar month, or the time in which the moon actually completes its revolution in the starry heaven, calculated on the average of 100 years, contains 27,321661 days. Thus, in 27321661 days, it would perform 1000000 revolutions. How does this relation, expressed by such large numbers, admit of being expressed by smaller ones?

Ans. The approximating values of 27,321661, are $\frac{27}{1}$, $\frac{82}{3}$, $\frac{765}{28}$, $\frac{3907}{143}$, &c; the third of which shows, that the moon revolves 28 times in 765 days, which differs from the truth only by a little more than 0,0001 of a day.

According to Laplace, one of the greatest Mathematicians of the last century, the siderial revolution of Mercury is 87,969255, and that of Venus 224,700817 days. How can these periods be represented by smaller numbers?

Ans. That of Mercury by $\frac{87}{1}$, $\frac{88}{1}$, $\frac{2815}{32}$, &c; and that of Venus by $\frac{224}{1}$, $\frac{225}{1}$, $\frac{674}{3}$, $\frac{1573}{7}$, $\frac{2247}{10}$, $\frac{26290}{117}$, &c. The fractions $\frac{2815}{32}$ and $\frac{26290}{117}$ give them already sufficiently exact.

The circumference of a circle is to its diameter as 3,1415926535 . . . . . to 1. How can this ratio be expressed in smaller numbers?

Ans. By 3 : 1, 22 : 7; 333 : 106; 355 : 113; 10393 : 33102; &c.

3) TRANSFORMATION OF $\sqrt{A}$ INTO A CONTINUED FRACTION.

It is assumed that A is a whole number. The following scheme shows the operation with regard to the rule given in VIII.

$$X = \sqrt{19} = 4 + \frac{\sqrt{19}-4}{1}\left(= \frac{1}{x}\right)$$

$$x = \frac{1}{\sqrt{19}-4} = \frac{\sqrt{19}+4}{3} = 2 + \frac{\sqrt{19}-2}{3}\left(= \frac{1}{x^{I}}\right)$$

$$x^{I} = \frac{3}{\sqrt{19}-2} = \frac{\sqrt{19}+2}{5} = 1 + \frac{\sqrt{19}-3}{5}\left(= \frac{1}{x^{II}}\right)$$

$$x^{II} = \frac{5}{\sqrt{19}-3} = \frac{\sqrt{19}+3}{2} = 3 + \frac{\sqrt{19}-3}{2}\left(= \frac{1}{x^{III}}\right)$$

$$x^{III} = \frac{2}{\sqrt{19}-3} = \frac{\sqrt{19}+3}{5} = 1 + \frac{\sqrt{19}-2}{5}\left(= \frac{1}{x^{IV}}\right)$$

$$x^{IV} = \frac{5}{\sqrt{19}-2} = \frac{\sqrt{19}+2}{3} = 2 + \frac{\sqrt{19}-4}{3}\left(= \frac{1}{x^{V}}\right)$$

$$x^{V} = \frac{3}{\sqrt{19}-4} = \frac{\sqrt{19}+4}{1} = 8 + \frac{\sqrt{19}-4}{1}\left(= \frac{1}{x^{VI}}\right)$$

$$x^{VI} = \frac{1}{\sqrt{19}-4} = \frac{\sqrt{19}+4}{3} = 2 + \&c.$$

The quotients are here 4, 2, 1, 3, 1, 2, 8, 2, &c.

EXAMPLES.

N	Given Roots.	Quotients.	Approximating Values.
1	$\sqrt{28}$	5, 3, 2, 3, 10, &c.	$\frac{5}{1}$, $\frac{16}{3}$, $\frac{37}{7}$, $\frac{127}{24}$, $\frac{1307}{247}$, &c.
2	$\sqrt{31}$	5, 1, 1, 3, 5, 3, 1, 1, 10, &c.	$\frac{5}{1}$, $\frac{6}{1}$, $\frac{11}{2}$, $\frac{39}{7}$, $\frac{206}{37}$, $\frac{657}{118}$, $\frac{863}{155}$, $\frac{1520}{273}$, $\frac{16063}{2885}$, &c.
3	$\sqrt{44}$	6, 1, 1, 1, 2, 1, 1, 1, 12, &c.	$\frac{6}{1}$, $\frac{7}{1}$, $\frac{13}{2}$, $\frac{20}{3}$, $\frac{53}{8}$, $\frac{73}{11}$, $\frac{126}{19}$, $\frac{199}{30}$, $\frac{2514}{379}$, &c.
4	$\sqrt{45}$	6, 1, 2, 2, 2, 1, 12, &c.	$\frac{6}{1}$, $\frac{7}{1}$, $\frac{20}{3}$, $\frac{47}{7}$, $\frac{114}{17}$, $\frac{161}{24}$, $\frac{2046}{305}$, &c.
5	$\sqrt{52}$	7, 4, 1, 2, 1, 4, 14, &c.	$\frac{7}{1}$, $\frac{29}{4}$, $\frac{36}{5}$, $\frac{101}{14}$, $\frac{137}{19}$, $\frac{649}{90}$, $\frac{9223}{1279}$, &c.
6	$\sqrt{53}$	7, 3, 1, 1, 3, 14, &c.	$\frac{7}{1}$, $\frac{22}{3}$, $\frac{29}{4}$, $\frac{51}{7}$, $\frac{182}{25}$, $\frac{2599}{357}$, &c.
7	$\sqrt{59}$	7, 1, 2, 7, 2, 1, 14, &c.	$\frac{7}{1}$, $\frac{8}{1}$, $\frac{23}{3}$, $\frac{169}{22}$, $\frac{361}{47}$, $\frac{530}{69}$, $\frac{7781}{1013}$, &c.
8	$\sqrt{67}$	8, 5, 2, 1, 1, 7, 1, 1, 2, 5, 16, &c.	$\frac{8}{1}$, $\frac{41}{5}$, $\frac{90}{11}$, $\frac{131}{16}$, $\frac{221}{27}$, $\frac{1678}{205}$, $\frac{1899}{232}$, $\frac{3577}{437}$, $\frac{9053}{1106}$, $\frac{48842}{5967}$, $\frac{790525}{96578}$, &c.

By an attentive examination of the quotients of $\sqrt{a}$, the following properties will be perceived.

1. The quotients recur in periods; consequently, in the above examples, it is only necessary to know the first period. The period begins with the second quotient, and ends with that, which is twice as great as the first.

2. Omitting the last quotient of the period, the others have the following order:

$$a, b, c, d, e \ldots\ldots\, e, d, c, b, a.$$

3. In general, let $\frac{p}{q}$ denote the approximating value belonging to the last quotient but one, of a period; then we have always

$$p^2 - Aq^2 = \pm 1.$$

The above examples afford sufficient illustration, at least for the first period. For we have

$$127^2 - 28 \,.\, 24^2 = +1, \quad 1520^2 - 31 \,.\, 273^2 = +1,$$
$$199^2 - 44 \,.\, 30^2 = +1, \quad 161^2 - 45 \,.\, 24^2 = +1,$$
$$649^2 - 52 \,.\, 90^2 = +1, \quad 182^2 - 53 \,.\, 25^2 = -1,$$
$$530^2 - 59 \,.\, 69^2 = +1, \quad 48842^2 - 67 \,.\, 5967^2 = +1.$$

4. When the period which belongs to the number A, consists of an even number of quotients, then we have for all periods $p^2 - Aq^2 = +1$; but when the number of quotients belonging to A, is odd; then $p^2 - Aq^2$ is alternately $= -1$ and $+1$.

5. In the operations which are necessary for determining the quotients, we always have integers to calculate with, and not fractions.

Continued fractions have besides the properties here mentioned, others no less remarkable. They are also valuable in practice. By them, for instance, (as we have already seen) we are enabled to express ratios which are given in large numbers, by smaller numbers, with the greatest accuracy. For it may be strictly proved, that when the two terms of a ratio are put one under the other in form of a fraction, and the approximate values of this fraction are sought, none of them can be written with less figures, or expressed in smaller numbers, than those obtained by continued fractions.

A small but very intelligible work on this subject is C. T. Kausler's Theory of Continued Fractions,* with their most important applications to Arithmetic and Algebra. Stuttgard. 1803. See also Legendre's Theory of Numbers,† where you will find the demonstrations of the last two theorems.

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* Die Lehre von den continuirlichen Bruechen nebst ihren vorzueglichsten Anwendungen auf Arithmetik und Algebra. Stuttgard. 1803.

† Théorie des Nombres, par Legendre.

SECTION II.

CONTAINING

ALGEBRAIC EQUATIONS.

XIII. THE ACCURATE SOLUTION OF ALGEBRAIC EQUATIONS, WITH SOME PRELIMINARY OBSERVATIONS.

1) EQUATIONS IN GENERAL.

I. An equation, in general, consists in the equality of two algebraic expressions, which are called its members. Every equation whose members are not *identical*, is either *analytical* or *algebraic.*

II. An analytic equation is one, in which the equality of the two members results solely from a proper interpretation of the signs and algebraic characters which it involves. Of this kind of equations are almost all those, which occur in the preceding section. If these equations involve letters, it is immaterial, what values they assume; the equation can be verified by the substitution of any numerical values.

III. An algebraic equation, on the contrary, is one which can be verified only by substituting, instead of the letters which it involves, certain determinate values (either numerical or algebraic.) Of this kind are those which occur in this section.

IV. Algebra is the science, by which magnitudes sought are obtained, from those that are known or given. Its object is, therefore, the solution of problems. The problem is first analysed, the relations between the required and given magnitudes are sought, and these relations are expressed by signs and letters, in the form of equations; distinguishing the known magnitudes from those which are unknown, only by a different notation. Thus, the last letters of the alphabet generally denote the quantities sought.

V. Modern Analysis treats only of the evolution of forms, and, therefore, only of analytic equations. It is, therefore, essentially different from Algebra, although, in their higher branches, they unite, and mutually assist each other. What is generally understood by the term Universal Arithmetic, embraces but the elements of this vast branch of mathematics.

VI. The *analysis* of the *ancients* is not essentially different from our Algebra; the latter having, however, the important advantage of a scientific Notation. The object of both is to find unknown quantities from given ones, by finding out their mutual relations, and expressing these in their simplest terms.

VII. The solution of equations is shown in most elementary treatises on Algebra. It may be remarked, however, for the sake of the beginner, that to determine the magnitudes required, from those that are given, there ought to be as many equations as unknown quantities; that, of these equations, none must be analytical; and that none of them must be so constituted as to be a necessary consequence of the others. Equations which do not answer the two last conditions, must be rejected as useless in the solution of Problems.

2) SIMPLE EQUATIONS.

a. WITH ONE UNKNOWN QUANTITY.

1) Equation $ax \pm b = c$*

Solution $x = \frac{c \mp b}{a}$

2) E. $3a + x - 5b + 2 = 7b - a + c + 6$

S. $x = 12b - 4a + c + 4$

3) E. $7 - 9a - 5x + 3cd + x = \frac{7}{4} - 3a - 2cd - 2x$

S. $x = \frac{21}{8} - 3a + \frac{5}{2} cd$

4) E. $8x - 5 = 13 - 7x$

S. $x = 1\frac{1}{5}$

5) E. $13\frac{3}{4} - \frac{x}{2} = 2x - 8\frac{3}{4}$

S. $x = 9$

6) E. $2x + 7 + \frac{3}{2}x = 6x - 23$

S. $x = 12$

7) E. $12\frac{1}{4} + 3x - 6 - \frac{7x}{3} = \frac{3x}{4} - 5\frac{3}{8}$

S. $x = 139\frac{1}{2}$

8) E. $-6\frac{1}{3}x + 158\frac{1}{2} - 10x = -\frac{37x}{6} + 19 + \frac{3}{8}x$

S. $x = 13\frac{59}{253}$

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* Hereafter the letter E. stands for *Equation*, and S. for *Solution*.

9) E. $8\frac{3}{4} + \frac{3x}{7} - \frac{5}{6} + 2x - \frac{12x}{5} + 13 + \frac{x}{4} = 0$
S. $x = -75\frac{10}{117}$

10) E. $\frac{3x}{5} - \frac{7x}{10} + \frac{3x}{4} - \frac{7x}{8} = -15$
S. $x = 66\frac{2}{3}$

11) E. $3\frac{2}{3} - x = \frac{9x}{2} + 8 = -17 - \frac{3x}{5} + \frac{3}{2}x$
S. $x = 4\frac{23}{48}$

12) E. $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} = 7x - 712 + \frac{x}{5}$
S. $x = 116\frac{148}{367}$

13) E. $11\frac{1}{2}x = \frac{11}{8}x + 66\frac{3}{8} - 5x - 9\frac{1}{4}$
S. $x = 3\frac{94}{121}$

14) E. $-\frac{16}{7}x = -\frac{1}{2}x + 412\frac{1}{2} - \frac{3}{5}x - 316\frac{1}{2}$
S. $x = -80\frac{80}{83}$

15) E. $32\frac{1}{10}x + 176\frac{3}{4} - x = 19\frac{1}{3}x + 7345 - \frac{2x}{3}$
S. $x = 576\frac{399}{746}$

16) E. $3{,}25x - 5{,}007 - x = 0{,}2 - 0{,}34x$
S. $x = 2{,}010424\ldots.$

17) E. $13{,}2 \,.\, x - \frac{3x}{4} + 7{,}6953 = \frac{x}{5} + 7834{,}5$
S. $x = 638{,}92283\ldots.$

18) E. $\frac{7{,}53x}{18} + 100 = \frac{2x}{5} + 3{,}86 - \frac{x}{6}$
S. $x = -519{,}67567\ldots,$

19) E. $ax + c = bx + d$

S. $x = \frac{d - c}{a - b}$

20) E. $\frac{f^2x}{cg} - \frac{a^2}{f} + cx = \frac{hx}{g} - c + (a + c)x$

S. $x = \frac{(a^2 - cf)cg}{(f^2 - ch - acg)f}$

21) E. $x = a + \frac{bc}{d} + \frac{cfx}{de}$

S. $x = \frac{(ad + bc)e}{de - cf}$

22) E. $\frac{ex}{f} + \frac{cx}{d} + \frac{ax}{b} - g = h$

S. $x = \frac{(h + g)bdf}{b(de + cf) + adf}$

23) E. $\frac{5}{6}ab + \frac{4}{5}ac - \frac{2}{3}cx = \frac{3}{4}ac + 2ab - 6cx$

S. $x = \frac{(70b - 3c)a}{320c}$

24) E. $\frac{x}{a} - 1 - \frac{dx}{c} + 3ab = 0$

S. $x = \frac{ac(1 - 3ab)}{c - ad}$

25) E. $\frac{ace}{d} - \frac{(a + b)^2x}{a} - bx = ae - 3bx$

S. $x = \frac{a^2e(c - d)}{(a^2 + b^2)d}$

26) E. $\frac{a + 3x}{4a} - \frac{7a - 5x}{6b} + 3 - \frac{9x}{4} = \frac{x}{ab} + \frac{5x}{6b}$

S. $x = \frac{39ab - 14a^2}{27ab - 9b + 12}$

27) E. $5a^3cx + ac^2x - 5abc^2 - 3a^3c^3 = 5a^2bcx + bc^2x$
$- 3a^2bc^3 - 5a^2c^2$

S. $x = \dfrac{3a^2c^2 - 5ac}{5a^2 + c}$

28) E. $2a^2b^2c + ab^2x - 2ab^3c - abc^2d - 3a^2x =$
$(b^3 - 3a^2b)x - b^2c^2d$

S. $x = \dfrac{2ab^2c - bc^2d}{3a^2 - b^2}$

29) E. $\dfrac{ab}{x} = bc + d + \dfrac{1}{x}$

S. $x = \dfrac{ab - 1}{bc + d}$

30) E. $\dfrac{3a + x}{x} - 5 = \dfrac{6}{x}$

S. $x = \dfrac{3a - 6}{4}$

31) E. $\dfrac{a^2x}{b - c} + dc = bx - ac$

S. $x = \dfrac{c(a + d)(b - c)}{b(b - c) - a^2}$

32) E. $c = a + \dfrac{m(a - x)}{3a + x}$

S. $x = \dfrac{a(m - 3c + 3a)}{c - a + m}$

33) E. $\dfrac{a(d^2 + x^2)}{dx} = ac + \dfrac{ax}{d}$

S. $x = \dfrac{d}{c}$

34) E. $\frac{cx^m}{a+bx} = \frac{fx^m}{d+ex}$

S. $x = \frac{cd-af}{bf-ce} = \frac{af-cd}{ce-bf}$

35) E. $\frac{7x^n}{x-1} = \frac{6x^{n+1}+x^n}{x+1} - \frac{3x^n+6x^{n+2}}{x^2-1}$

S. $x = -\frac{11}{12}$

36) E. $\frac{3a-5x}{a-c} + \frac{2a-x}{d} = \frac{a+f}{a-c} - dx$

S. $x = \frac{d(f-2a)-2a(a-c)}{(a-c)(d^2-1)-5d}$

37) E. $\frac{a}{bx} + \frac{c}{dx} + \frac{e}{fx} + \frac{g}{hx} = k$

S. $x = \frac{adfh+bcfh+bdeh+bdfg}{bdfhk}$

38) E. $\frac{3abc}{a+b} + \frac{a^2b^2}{(a+b)^3} + \frac{(2a+b)b^2x}{a(a+b)^2} = 3cx + \frac{bx}{a}$

S. $x = \frac{ab}{a+b}$

39) E. $\frac{bx}{2b-a} - \frac{(3bc+ad)x}{2ab(a+b)} - \frac{5ab}{3c-d} = \frac{(3bc-ad)x}{2ab(a-b)}$
$- \frac{5a(2b-a)}{a^2-b^2}$

S. $x = \frac{5a(2b-a)}{3c-d}$

40) E. $(a+x)(b+x) - a(b+c) = \frac{a^2c}{b} + x^2$

S. $x = \frac{ac}{b}$

41) E. $\sqrt[m]{x} = a$

S. $x = a^m$

42) E. $\sqrt[m]{(ax + b)} = \sqrt[m]{(cx + d)}$

S. $x = \frac{d - b}{a - c}$

43) E. $h\sqrt[3]{(ax - b)} = k\sqrt[3]{(cx + dx - f)}$

S. $x = \frac{bh^3 - fk^3}{ah^3 - (c + d)k^3}$

44) E. $\sqrt[3]{(a^2 + c)} = \sqrt[4]{\frac{a^2 + c}{d(x + g)}}$

S. $x = \frac{1}{d\sqrt[3]{(a^2 + c)}} - g$

45) E. $\sqrt[m]{(a + x)} = \sqrt[2m]{(x^2 + 5ax + b^2)}$

S. $x = \frac{a^2 - b^2}{3a}$

46) E. $c + b\sqrt[m]{(x + d)} = f$

S. $x = \left(\frac{f - c}{b}\right)^m - d$

47) E. $\frac{ax}{b}\sqrt{(f^2x^2 + d^2)} + \frac{afx^2}{b} = cx$

S. $x = \frac{b^2c^2 - a^2d^2}{2abcf} = \frac{(bc + ad)\,(bc - ad)}{2abcf}$

LOGARITHMIC EQUATIONS.

48) E. $a^x = b$

S. $x = \dfrac{\log b}{\log a}$

49) E. $a^{mx}b^{nx} = c$

S. $x = \dfrac{\log c}{m \log a + n \log b} = \dfrac{\log c}{\log a^m b^n}$

50) E. $a^{mx+f}b^{nx+g} = c^{px+h}d^{qx+k}$

S. $x = \dfrac{h \log c + k \log d - f \log a - g \log b}{m \log a + n \log b - p \log c - q \log d}$

$= \left(\log \dfrac{c^h d^k}{a^f b^g}\right) : \left(\log \dfrac{a^m b^n}{c^p d^q}\right)$

51) E. $3^x = 177147$

S. $x = 11$

52) E. $2^x = 769$

S. $x = 9{,}586839 \ldots.$

53) E. $\left(\dfrac{3}{4}\right)^x = 51\tfrac{1}{2}$

S. $= - 13{,}701172 \ldots.$

54) E. $\left(\dfrac{756}{345}\right)^{\frac{3x}{2}} = 54783$

S. $x = 9{,}272299 \ldots.$

55) E. $\left(\dfrac{21}{20}\right)^x \left(\dfrac{3}{5}\right)^{\frac{7x}{2}} = \tfrac{7}{12}$

S. $x = 0{,}309928 \ldots.$

56) E. $\left(\frac{295}{867}\right)^{3-x} = 632.\left(\frac{56}{39}\right)^{\frac{5x}{9}}$

S. $x = 11{,}040270 \ldots.$

57) E. $3^{2x}.5^{6x-7} = 9^{x-2}.7^{1-x}$

S. $x = 0{,}759965 \ldots.$

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b. WITH SEVERAL UNKNOWN QUANTITIES.

1) E. $\left\{\begin{array}{l} x + y = a \\ x - y = b \end{array}\right\}$

S. $x = \frac{a+b}{2}$, $y = \frac{a-b}{2}$

2) E. $\left\{\begin{array}{r} 3x + 2y = 118 \\ x + 5y = 191 \end{array}\right\}$

S. $x = 16$, $y = 35$

3) E. $\left\{\begin{array}{l} 7x + \frac{5}{2}y = 411\frac{1}{2} \\ 39x - 14y = -935\frac{9}{10} \end{array}\right\}$

S. $x = 17\frac{1}{2}$, $y = 115\frac{3}{5}$

4) E. $\left\{\begin{array}{l} 5x - 8\frac{1}{2} = 7y - 44 \\ 2x = y + \frac{5}{7} \end{array}\right\}$

S. $x = 4\frac{1}{2}$, $y = 8\frac{2}{7}$

5) E. $\left\{\begin{array}{l} 5\frac{3}{4}y - 11x = 4y + 117\frac{1}{8} \\ 8x + 175 = 2y \end{array}\right\}$

S. $x = 9$, $y = 123\frac{1}{2}$

6) E. $\left\{\begin{array}{l} 7y = 2x - 3y \\ 19x = 60y + 621\frac{1}{4} \end{array}\right\}$

S. $x = 88\frac{3}{4}, \quad y = 17\frac{3}{4}$

7) E. $\left\{\begin{array}{l} 13x + 7y - 341 = 7\frac{1}{2}y + 43\frac{1}{2}x \\ 2x + \frac{1}{2}y = 1 \end{array}\right\}$

S. $x = -12, \quad y = 50$

8) E. $\left\{\begin{array}{l} 113\frac{1}{2}x - 27\frac{5}{7}y = 10y + 5488\frac{4}{7} \\ 9y - 347 = 5x - 420 \end{array}\right\}$

S. $x = 56, \quad y = 23$

9) E. $\left\{\begin{array}{l} 168\frac{15}{44} - 19x + \frac{3}{11}y = 12\frac{3}{4}x = 1084 \\ \frac{1}{3}x - 149\frac{1}{2} = 319\frac{2}{9} - \frac{7}{2}y \end{array}\right\}$

S. $x = -27\frac{2}{3}, \quad y = 136\frac{5}{9}$

10) E. $\left\{\begin{array}{l} x + y = 18{,}73 \\ 0{,}56x + 13{,}421y = 763{,}4 \end{array}\right\}$

S. $x = -39{,}8121\ldots, \quad y = 58{,}5421\ldots$

11) E. $\left\{\begin{array}{l} (x + 5)(y + 7) = (x + 1)(y - 9) + 112 \\ 2x + 10 = 3y + 1 \end{array}\right\}$

S. $x = 3, \quad y = 5$

12) E. $\left\{\begin{array}{l} ax = by \\ x + y = c \end{array}\right\}$

S. $x = \frac{bc}{a + b}, \quad y = \frac{ac}{a + b}$

13) E. $\left\{\begin{array}{l} ax + by = c \\ fx + gy = h \end{array}\right\}$

S. $x = \frac{cg - bh}{ag - bf}, \quad y = \frac{ah - cf}{ag - bf}$

14) E. $\left\{\begin{array}{l} \dfrac{a}{b+y} = \dfrac{b}{3a+x} \\ ax + 2by = d \end{array}\right\}$

S. $x = \dfrac{2b^2 - 6a^2 + d}{3a}$, $y = \dfrac{3a^2 - b^2 + d}{3b}$

15) E. $\left\{\begin{array}{l} bcx = cy - 2b \\ b^2y + \dfrac{a(c^3 - b^3)}{bc} = \dfrac{2b^3}{c} + c^3x \end{array}\right\}$

S. $x = \dfrac{a}{bc}$, $y = \dfrac{a + 2b}{c}$

16) E. $\left\{\begin{array}{l} 3x + 5y = \dfrac{(8b - 2f)bf}{b^2 - f^2} \\ b^2x - \dfrac{bcf^2}{b+f} + (b+c+f)fy = f^2x + (b+2f)bf \end{array}\right\}$

S. $x = \dfrac{bf}{b-f}$, $y = \dfrac{bf}{b+f}$

17) E. $\left\{\begin{array}{l} x + y = 10 \\ x + z = 19 \\ y + z = 23 \end{array}\right\}$

S. $x = 3$, $y = 7$, $z = 16$

18) E. $\left\{\begin{array}{l} x + y + z = 29\frac{1}{4} \\ x + y - z = 18\frac{1}{4} \\ x - y + z = 13\frac{3}{4} \end{array}\right\}$

S. $x = 16$, $y = 7\frac{3}{4}$, $z = 5\frac{1}{2}$

19) E. $$\left\{\begin{array}{l} x + y + z = a \\ my = nx \\ pz = qx \end{array}\right\}$$

S. $$x = \frac{amp}{mp + np + mq},\quad y = \frac{anp}{mp + np + mq},$$
$$z = \frac{amq}{mp + np + mq}$$

20) E. $$\left\{\begin{array}{l} 3x + 5y = 161 \\ 7x + 2z = 209 \\ 2y + z = 89 \end{array}\right\}$$

S. $x = 17,\ y = 22,\ z = 45$

21) E. $$\left\{\begin{array}{l} y + \frac{1}{2}x = 41 \\ x + \frac{1}{4}z = 20\frac{1}{2} \\ y + \frac{1}{5}z = 34 \end{array}\right\}$$

S. $x = 18,\ y = 32,\ z = 10$

22) E. $$\left\{\begin{array}{l} ax + by = c \\ dx + ey = f \\ gy + hz = l \end{array}\right\}$$

S. $$x = \frac{ce - bf}{ae - bd},\quad y = \frac{af - cd}{ae - bd},\quad z = \frac{a(el - fg) - d(bl - cg)}{h(ae - bd)}$$

23) E. $$\left\{\begin{array}{r} 53 - \frac{1}{2}x - \frac{1}{2}z = y - 109 \\ \frac{1}{4}x + \frac{1}{8}y = 26 \\ 5y = 4z \end{array}\right\}$$

S. $x = 64,\ y = 80,\ z = 100$

24) E. $$\left\{\begin{array}{l} 2x - \frac{3}{4}y = 93 - \frac{1}{2}x - \frac{1}{4}y \\ 7x - 5z = y + x - 86 \\ \frac{1}{2}x + \frac{1}{3}y + \frac{1}{4}z = 58 \end{array}\right\}$$

S. $x = 48,\ y = 54,\ z = 64$

25) E. $\left\{\begin{array}{l} 3x - 100 = 5y + 360 \\ 2\frac{1}{3}x + 200 = 16\frac{1}{2}z - 610 \\ 2y + 3z = 548 \end{array}\right\}$

S. $x = 360,\ y = 124,\ z = 100$

26) E. $\left\{\begin{array}{l} 4x + 7y + 159 = 0 \\ 3\frac{1}{3}x = \frac{1}{6}z - 55 \\ 2x + y + 9z = 498 \end{array}\right\}$

S. $x = -13\frac{1}{2},\ y = -15,\ z = 60$

27) E. $\left\{\begin{array}{l} 2x + 5y - 7z = -288 \\ 5x - y + 3z = 227 \\ 7x + 6y + z = 297 \end{array}\right\}$

S. $x = 13,\ y = 24,\ z = 62$

28) E. $\left\{\begin{array}{r} x + y + z = 30 \\ 8x + 4y + 2z = 50 \\ 27x + 9y + 3z = 64 \end{array}\right\}$

S. $x = \frac{2}{3},\ y = -7,\ z = 36\frac{1}{3}$

29) E. $\left\{\begin{array}{l} 18x - 7y - 5z = 11 \\ 4\frac{2}{3}y - \frac{2}{3}x + z = 108 \\ 3\frac{1}{2}z + 2y + \frac{3}{4}x = 80 \end{array}\right\}$

S. $x = 12,\ y = 25,\ z = 6$

30) E. $\left\{\begin{array}{l} ax + by + cz = h \\ a'x + b'y + c'z = h' \\ a''x + b''y + c''z = h'' \end{array}\right\}$

S. $x = \dfrac{hb'c'' - hb''c' + h'b''c - h'bc'' + h''bc' - h''b'c}{ab'c'' - ab''c' + a'b''c - a'bc'' + a''bc' - a''b'c}$

$y = \dfrac{ah'c'' - ah''c' + a'h''c - a'hc'' + a''hc' - a''h'c}{ab'c'' - ab''c' + a'b''c - a'bc'' + a''bc' - a''b'c}$

$z = \dfrac{ab'h'' - ab''h' + a'b''h - a'bh'' + a''bh' - a''b'h}{ab'c'' - ab''c' + a'b''c - a'bc'' + a''bc' - a''b'c}$

31) E. $$\left\{\begin{array}{l} \frac{1}{x}+\frac{1}{y}=a \\ \frac{1}{x}+\frac{1}{z}=b \\ \frac{1}{y}+\frac{1}{z}=c \end{array}\right\}$$

S. $x=\frac{2}{a+b-c}$, $y=\frac{2}{a-b+c}$, $z=\frac{2}{b+c-a}$

32) E. $$\left\{\begin{array}{l} \frac{2}{x}-\frac{5}{3y}+\frac{1}{z}=3\tfrac{4}{27} \\ \frac{1}{4x}+\frac{1}{y}+\frac{2}{z}=6\tfrac{11}{72} \\ \frac{5}{6x}-\frac{1}{y}+\frac{4}{z}=12\tfrac{1}{36} \end{array}\right\}$$

S. $x=6$, $y=9$, $z=\frac{1}{3}$

33) E. $$\left\{\begin{array}{l} \frac{2x+3y}{x+y}=2\tfrac{1}{5} \\ \frac{x+z}{5(x-z)}=\frac{1}{3} \\ \frac{10x-3z}{4x-2z}=2\tfrac{9}{14} \end{array}\right\}$$

S. $x=16$, $y=4$, $z=4$; and others.
(Undetermined.)

34) E. $$\left\{\begin{array}{l} ax+by=l \\ cx+du=m \\ ex+fz=n \\ gy+hz=p \end{array}\right\}$$

$$\text{S. } x = \frac{bhn + (gl - bp)f}{beh + afg}$$

$$y = \frac{afp + (el - an)h}{beh + afg}$$

$$z = \frac{bep + (an - el)g}{beh + afg}$$

$$u = \frac{bh(em - cn) + gf(am - cl) + bcfp}{d(beh + afg)}$$

35) E. $$\left\{\begin{array}{r} x - 9y + 3z - 10u = 21 \\ 2x + 7y - z - u = 683 \\ 3x + y + 5z + 2u = 195 \\ 4x - 6y - 2z + 9u = 516 \end{array}\right\}$$

S. $x = 100,\ y = 60,\ z = -13,\ u = -50$

36) E. $$\left\{\begin{array}{r} x + y + z + u = 1 \\ 16x + 8y + 4z + 2u = 9 \\ 81x + 27y + 9z + 3u = 36 \\ 256x + 64y + 16z + 4u = 100 \end{array}\right\}$$

S. $x = \frac{1}{4},\ y = \frac{1}{2},\ z = \frac{1}{4},\ u = 0$

37) E. $$\left\{\begin{array}{r} \frac{x}{3} + \frac{y}{5} + \frac{2z}{7} = 58 \\ \frac{5x}{4} + \frac{y}{6} + \frac{z}{3} = 76 \\ \frac{x}{2} + \frac{3z}{8} + \frac{u}{5} = 79 \\ y + z + u = 248 \end{array}\right\}$$

S. $x = 12,\ y = 30,\ z = 168,\ u = 50$

38) E. $$\left\{\begin{array}{l} \dfrac{xy}{ay+bx}=l \\ \dfrac{yz}{cz+dy}=m \\ \dfrac{xz}{ez+fx}=n \end{array}\right\}$$

S. $$x=\frac{lmn(bde+acf)}{cfmn-bfln+bdlm}$$

$$y=\frac{lmn(bde+acf)}{afln+demn-ad}$$

$$z=\frac{lmn(bde+acf)}{beln-cemn+aclm}$$

39) E. $$\left\{\begin{array}{l} x+y+z+t+u=a \\ x+y+z+u+w=b \\ x+y+z+t+w=c \\ x+y+u+t+w=d \\ x+z+u+t+w=e \\ y+z+u+t+w=f \end{array}\right\}$$

S. $$x=\frac{s}{5}-f,\quad y=\frac{s}{5}-e,\quad z=\frac{s}{5}-d,$$

$$u=\frac{s}{5}-c,\quad t=\frac{s}{5}-b,\quad w=\frac{s}{5}-a,$$

where $(a+b+c+d+e+f=s)$

3) QUADRATIC EQUATIONS.

a. WITH ONE UNKNOWN QUANTITY.

FORMULÆ.

E. $x^2 + Px = Q$

S. $x = -\frac{P}{2} \pm \sqrt{\left(\frac{P^2}{4} + Q\right)}$

EXAMPLES.

1) E. $ax^2 = b$

S. $x = +\sqrt{\frac{b}{a}}, \quad x = -\sqrt{\frac{b}{a}}$

2) E. $x^2 + 6x = 27$

S. $x = 3, \quad x = -9$

3) E. $x^2 - 7x + 3\frac{1}{4} = 0$

S. $x = 6\frac{1}{2}, \quad x = \frac{1}{2}$

4) E. $x^2 - 5\frac{3}{4}x = 18$

S. $x = 8, \quad x = -2\frac{1}{4}$

5) E. $3x^2 - 2x = 65$

S. $x = 5, \quad x = -4\frac{1}{3}$

6) E. $622x = 15x^2 + 6384$

S. $x = 22\frac{4}{5}, \quad x = 18\frac{2}{3}$

7) E. $20748 - 1616x + 21x^2 = 0$

S. $x = 60\frac{2}{3}, \quad x = 16\frac{2}{7}$

9) E. $11\frac{3}{4}x - 3\frac{1}{2}x^2 = -41\frac{1}{4}$

S. $x = -2\frac{1}{7}$, $x = 5\frac{1}{2}$

10) E. $9\frac{1}{3}x^2 - 90\frac{1}{3}x + 195 = 0$

S. $x = 6\frac{3}{7}$, $x = 3\frac{1}{4}$

11) E. $\frac{18x^2}{5} + \frac{18078x}{65} + 4728 = 0$

S. $x = -25\frac{10}{39}$, $x = -52$

12) E. $x^2 - 8x = 14$

S. $x = 4 + \sqrt{30}$, $x = 4 - \sqrt{30}$

or $x = 9{,}4772\ldots.$, $x = -1{,}4772\ldots.$

13) E. $3x^2 + x = 7$

S. $x = \frac{-1 + \sqrt{85}}{6}$, $x = \frac{-1 - \sqrt{85}}{6}$

or $x = 1{,}3699\ldots$, $x = -1{,}7032\ldots$

14) E. $118x - 2\frac{1}{2}x^2 = 20$

S. $x = \frac{118 + \sqrt{13724}}{5}$, $x = \frac{118 - \sqrt{13724}}{5}$

or $x = 47{,}0298\ldots$, $x = 0{,}1701\ldots$

15) E. $6x - 30 = 3x^2$

S. $x = 1 + \sqrt{-9}$, $x = 1 - \sqrt{-9}$

16) E. $8x^2 - 7x + 34 = 0$

S. $x = \frac{7 + \sqrt{-1039}}{16}$, $x = \frac{7 - \sqrt{-1039}}{16}$

17) E. $4x^2 - 9x = 5x^2 - 255\frac{3}{4} - 8x$

S. $x = 15\frac{1}{2}$, $x = -16\frac{1}{2}$

18) E. $80x + \frac{3x^2}{4} + \frac{21x - 27782}{12} = 1859\frac{1}{3} - 3x^2$

S. $x = -46, \ x = 24\frac{1}{5}$

19) E. $\frac{x}{x + 60} = \frac{7}{3x - 5}$

S. $x = 14, \ x = -10$

20) E. $\frac{40}{x - 5} + \frac{27}{x} = 13$

S. $x = 9, \ x = 1\frac{2}{13}$

21) E. $\frac{8x}{x + 2} - 6 = \frac{20}{3x}$

S. $x = 10, \ x = -\frac{2}{3}$

22) E. $\frac{48}{x + 3} = \frac{165}{x + 10} - 5$

S. $x = 5\frac{2}{5}, \ x = 5$

23) E. $\frac{31}{6x} = \frac{16}{117 - 2x} + 1$

S. $x = 67\frac{1}{6}, \ x = 4\frac{1}{2}$

24) E. $\frac{2x + 3}{10 - x} = \frac{2x}{25 - 3x} - 6\frac{1}{2}$

S. $x = 13\frac{32}{31}, \ x = 8$

25) E. $\frac{25x + 180}{10x - 81} = \frac{40x}{5x - 8} - \frac{3}{5}$

S. $x = 14\frac{3}{5}, \ x = \frac{72}{245}$

26) E. $\frac{18 + x}{6(3 - x)} = \frac{20x + 9}{19 - 7x} - \frac{65}{4(3 - x)}$

S. $x = 7\frac{22}{113} \ \ x = 2\frac{1}{2}$

27) E. $adx - acx^2 = bcx - bd$

S. $x = \frac{d}{c}$, $x = -\frac{b}{a}$

28) E. $\frac{a^2x^2}{f^2} - \frac{2ax}{g} + \frac{f^2}{g^2} = 0$

S. $x = \frac{f^2}{ag}$, $x = \frac{f^2}{ag}$

29) E. $abx^2 + \frac{3a^2x}{c} = \frac{6a^2 + ab - 2b^2}{c^2} - \frac{b^2x}{c}$

S. $x = \frac{2a - b}{ac}$, $x = -\frac{3a + 2b}{bc}$

30) E. $\frac{2c^2}{d^2} + \frac{ac}{d} - (a - b)(2c + ad)\frac{x}{d} = (a + b)\frac{cx}{d}$
$- (a^2 - b^2)x^2$

S. $x = \frac{2c + ad}{d(a + b)}$, $x = \frac{c}{d(a - b)}$

31) E. $32a^{2m}c^{n-1} + 4a^{m+3}c^{n-1}(ac^3 - 2)x = a^7c^{n+2}x^2$

S. $x = 4a^{m-3}$, $x = -\frac{8a^{m-4}}{c^3}$

32) E. $cx + \frac{ac}{a + b} = (a + b)x^2$

S. $x = \frac{c + \sqrt{(c^2 + 4ac)}}{2(a + b)}$, $x = \frac{c - \sqrt{(c^2 + 4ac)}}{2(a + b)}$

33) E. $9a^4b^4x^2 - 6a^3b^2x - b^2 = 0$

S. $x = \frac{a + \sqrt{(a^2 + b^2)}}{3a^2b^2}$, $x = \frac{a - \sqrt{(a^2 + b^2)}}{3a^2b^2}$

34) E. $abx^2 - 2x(a + b)\sqrt{ab} = (a - b)^2$

S. $x = \frac{a + b \pm \sqrt{(2a^2 + 2b^2)}}{\sqrt{ab}}$

35) E. $ax^2 + b^2 + c^2 = a^2 + 2bc + 2(b - c)x\sqrt{a}$

S. $x = \frac{b - c + a}{\sqrt{a}}$, $x = \frac{b - c - a}{\sqrt{a}}$

36) E. $cx^2 - 2cx\sqrt{d} = dx^2 - cd$

S. $x = \frac{\sqrt{cd}}{\sqrt{c} + \sqrt{d}}$, $x = \frac{\sqrt{cd}}{\sqrt{c} - \sqrt{d}}$

37) E. $(4a^2 - 9cd^2)x^2 + (4a^2c^2 + 4abd^2)x +$
$(ac^2 + bd^2)^2 = 0$

S. $x = -\frac{ac^2 + bd^2}{2a + 3d\sqrt{c}}$, $x = -\frac{ac^2 + bd^2}{2a - 3d\sqrt{c}}$

38) E. $ab^3x^2 + (1 + c)bd\sqrt{c} + cb^2x^2 =$
$[b^3d\sqrt{c} + (ab + c)(1 + c)]x$

S. $x = \frac{bd\sqrt{c}}{ab + c}$, $x = \frac{1 + c}{b^2}$

39) E. $\frac{5a + 10ab^2}{9b^2 - 3a^2b^2}x^2 - \left(\frac{5\sqrt{(a + b)}}{3b^3} + \frac{(1 + 2b^2)cd\sqrt{c}}{3 - a^2}\right)x$
$+ \frac{cd}{ab}\sqrt{(a + b)c} = 0$

S. $x = \frac{(3 - a^2)\sqrt{(a + b)}}{ab(1 + 2b^2)}$, $x = \frac{3b^2cd\sqrt{c}}{5a}$

40) E. $ax = b + \sqrt{cx}$

S. $x = \frac{2ab + c + \sqrt{(4abc + c^2)}}{2a^2}$*

---

* Only this value of x can here be used; the other satisfies the equation $ax = b - \sqrt{cx}$; because this equation and the given one lead to the same final equation $a^2x^2 - (2ab + c)x + b^2 = 0$. These observations apply also to the equations in No. 41, 42, 43, 44.

41) E. $3\sqrt{(112-8x)} = 19 + \sqrt{(3x+7)}$

S. $x = 6$

42) E. $\sqrt{(2x+7)} + \sqrt{(3x-18)} = \sqrt{(7x+1)}$

S. $x = 9$

43) E. $5\sqrt{62+3x)} - \tfrac{1}{2}\sqrt{(95\tfrac{2}{3}-5x)} = 41$

S. $x = 6\tfrac{1}{3}$

44) E. $7\sqrt{(\tfrac{3}{2}x-5)} - \sqrt{\left(\frac{x}{5}+45\right)} - \tfrac{7}{4}\sqrt{(10x+56)} = 0$

S. $x = 20$

45) E. $ax^{2n} + bx^{n} = c$

S. $x = \sqrt[n]{\dfrac{-b+\sqrt{(b^2+4ac)}}{2a}},$

$x = \sqrt[n]{\dfrac{-b-\sqrt{(b^2+4ac)}}{2a}}$

46) E. $x^4 - 74x^2 = -1225$

S. $x = \pm 5,\ \ x = \pm 7$

47) E. $3x^6 + 42x^3 = 3321$

S. $x = 3,\ \ x = -\sqrt[3]{41}$

---

b. WITH SEVERAL UNKNOWN QUANTITIES.

1) E. $\begin{Bmatrix} x^2 + y^2 = a \\ x^2 - y^2 = b \end{Bmatrix}$

S. $x = \pm\sqrt{\frac{a+b}{2}},\ y = \pm\sqrt{\frac{a-b}{2}}$

2) E. $\left\{\begin{aligned} x+y&=a \\ xy&=b \end{aligned}\right\}$

S. $x=\frac{a+\sqrt{(a^2-4b)}}{2},\ y=\frac{a-\sqrt{(a^2-4b)}}{2}$

or $x=\frac{a-\sqrt{(a^2-4b)}}{2},\ y=\frac{a+\sqrt{(a^2-4b)}}{2}$

3) E. $\left\{\begin{aligned} x+y&=a \\ x^2+y^2&=b \end{aligned}\right\}$

S. $x=\frac{a+\sqrt{(2b-a^2)}}{2},\ y=\frac{a-\sqrt{(2b-a^2)}}{2}$

or $x=\frac{a-\sqrt{(2b-a^2)}}{2},\ y=\frac{a+\sqrt{(2b-a^2)}}{2}$

4) E. $\left\{\begin{aligned} xy&=a \\ x^2+y^2&=b \end{aligned}\right\}$

S. $x=\pm\sqrt{\frac{b+\sqrt{(b^2-4a^2)}}{2}}$

$y=\pm\sqrt{\frac{b-\sqrt{(b^2-4a^2)}}{2}}$

or $x=\pm\sqrt{\frac{b-\sqrt{(b^2-4a^2)}}{2}}$

$y=\pm\sqrt{\frac{b+\sqrt{(b^2-4a^2)}}{2}}$

5) E. $\left\{\begin{aligned} x+y&=a \\ x^3+y^3&=b \end{aligned}\right\}$

S. $x=\frac{a}{2}+\sqrt{\frac{4b-a^3}{12a}},\ y=\frac{a}{2}-\sqrt{\frac{4b-a^3}{12a}}$

or $x=\frac{a}{2}-\sqrt{\frac{4b-a^3}{12a}},\ y=\frac{a}{2}+\sqrt{\frac{4b-a^3}{12a}}$

6) E. $\begin{Bmatrix} 2x + 3y = 118 \\ 5x^2 - 7y^2 = 4333 \end{Bmatrix}$

S. $x = 35, \ y = 16$

or $x = -229\tfrac{6}{17}, \ y = 192\tfrac{4}{17}$

7) E. $\begin{Bmatrix} ax + by = h \\ cx^2 + dy^2 = k \end{Bmatrix}$

$$\text{S.} \quad x = \frac{adh \pm b\sqrt{(a^2dk - cdh^2 + b^2ck)}}{a^2d + b^2c}$$

$$y = \frac{bch \pm a\sqrt{(a^2dk - cdh^2 + b^2ck)}}{a^2d + b^2c}$$

8) E. $\begin{Bmatrix} x^2 - y^2 = h \\ (x + y + a)^2 + (x - y + a)^2 = k \end{Bmatrix}$

$$\text{S.} \quad x = \frac{-a \pm \sqrt{(2h + k - a^2)}}{2}$$

$$y = \pm\sqrt{\frac{\tfrac{1}{2}k - h \mp a\sqrt{(2h + k - a^2)}}{2}}$$

9) E. $\left\{ \begin{array}{l} \dfrac{18x}{y} = \dfrac{8y}{x} \\ 3xy + 2x + y = 485 \end{array} \right\}$

S. $x = 10, \ y = 15$

or $x = -10\tfrac{7}{9}, \ y = -16\tfrac{1}{6}$ *

---

* These equations admit of two more solutions, viz.

$$x = \frac{1 + \sqrt{-34919}}{18}, \ y = \frac{-1 - \sqrt{-34919}}{12} \text{ and}$$

$$x = \frac{1 - \sqrt{-34919}}{18}, \ y = \frac{-1 + \sqrt{-34919}}{12}$$

which, however, are imaginary.

10) E. $\left\{\begin{array}{l} \dfrac{ax}{y} = \dfrac{by}{x} \\ cxy + dx + ey = h \end{array}\right\}$

$$*\ \text{S.}\left\{\begin{array}{l} x = \dfrac{-(e\sqrt{a}+d\sqrt{b}) \pm \sqrt{[(e\sqrt{a}+d\sqrt{b})^2+4ch\sqrt{ab}]}}{2c\sqrt{a}} \\ y = \dfrac{-(e\sqrt{a}+d\sqrt{b}) \pm \sqrt{[(e\sqrt{a}+d\sqrt{b})^2+4ch\sqrt{ab}]}}{2c\sqrt{a}} \end{array}\right\}$$

$$\text{or}\left\{\begin{array}{l} x = \dfrac{-(e\sqrt{a}-d\sqrt{b}) \pm \sqrt{[(e\sqrt{a}-d\sqrt{b})^2-4ch\sqrt{ab}]}}{2c\sqrt{b}} \\ y = \dfrac{+(e\sqrt{a}-d\sqrt{b}) \mp \sqrt{[(e\sqrt{a}-d\sqrt{b})^2-4ch\sqrt{ab}]}}{2c\sqrt{b}} \end{array}\right\}$$

11) E. $\left\{\begin{array}{l} x + y + x^2 + y^2 = a \\ x - y + x^2 - y^2 = b \end{array}\right\}$

$$\text{S. } x = \frac{-1 \pm \sqrt{(2a + 2b + 1)}}{2}$$

$$y = \frac{-1 \pm \sqrt{(2a - 2b + 1)}}{2}$$

$$\text{or } x = \frac{-1 \pm \sqrt{(2a + 2b + 1)}}{2}$$

$$y = \frac{-1 \mp \sqrt{(2a - 2b + 1)}}{2}\ \dagger$$

12) E. $\left\{\begin{array}{l} x + y = xy \\ x + y + x^2 + y^2 = a \end{array}\right\}$

---

* These equations give four values for x and y, of which two and two correspond with each other.

† These equations give again four values for x and y, of which two and two correspond with each other.

S. $$x = \frac{1 \pm \sqrt{(4a+1)} + \sqrt{[4a - 6 \mp 6\sqrt{(4a+1)}]}}{4}$$

$$y = \frac{1 \pm \sqrt{(4a+1)} - \sqrt{[4a - 6 \mp 6\sqrt{(4a+1)}]}}{4}\,*$$

13) E. $$\left\{\begin{array}{l} ax - by = g \\ a^3x^3 - b^3y^3 = hxy \end{array}\right\}$$

S. $$x = \frac{g}{2a}\left(1 \pm \sqrt{\frac{h + abg}{h - 3abg}}\right)$$

$$y = \frac{g}{2b}\left(-1 \pm \sqrt{\frac{h + abg}{h - 3abg}}\right)$$

14) E. $$\left\{\begin{array}{l} (x - y)\,(x^2 - y^2) = a \\ (x + y)\,(x^2 + y^2) = b \end{array}\right\}$$

S. $$x = \frac{\sqrt{(2b - a)} \pm \sqrt{a}}{2\sqrt[6]{(2b - a)}},\quad y = \frac{\sqrt{(2b - a)} \mp \sqrt{a}}{2\sqrt[6]{(2b - a)}}$$

15) E. $$\left\{\begin{array}{l} x^2 + y^2 + z^2 = a \\ y^2 = 2xz + b \\ cx = dz \end{array}\right\}$$

S. $$x = \frac{d\sqrt{(a - b)}}{c + d},\quad y = \frac{\sqrt{[2acd + b(c^2 + d^2)]}}{c + d},$$

$$z = \frac{c\sqrt{(a - b)}}{c + d}$$

16) E. $$\left\{\begin{array}{l} x(y + z) = a \\ y(x + z) = b \\ z(x + y) = c \end{array}\right\}$$

---

* The values of x and y may be interchanged.

$$\text{S. } x = \pm\sqrt{\frac{(a-c+b)\,(a-b+c)}{2(c-a+b)}}$$

$$y = \pm\sqrt{\frac{(a-c+b)\,(c-a+b)}{2(a-b+c)}}$$

$$z = \pm\sqrt{\frac{(c-a+b)\,(a-b+c)}{2(a-c+b)}}$$

$$\text{17) E. } \left\{\begin{array}{l} \dfrac{xyz}{x+y} = a \\ \dfrac{xyz}{y+z} = b \\ \dfrac{xyz}{x+z} = c \end{array}\right\}$$

$$\text{S. } x = \pm\sqrt{\frac{2abc(ab-ac+bc)}{(ab+ac-bc)\,(bc+ac-ab)}}$$

$$y = \pm\sqrt{\frac{2abc(bc+ac-ab)}{(ab+ac-bc)\,(ab-ac+bc)}}$$

$$z = \pm\sqrt{\frac{2abc(ab+ac-bc)}{(ab-ac+bc)\,(bc+ac-ab)}}$$

$$\text{18) E. } \left\{\begin{array}{l} xy = p \\ (b-y)z = p' \\ (a-x)\;(c-z) = p'' \end{array}\right\}$$

$$\text{S. } x = \frac{-A \pm\sqrt{[A^2 - 4p(p'-bc)\,(p''-ac)]}}{2(p'-bc)}$$

$$y = \frac{-A \mp\sqrt{[A^2 - 4p(p'-bc)\,(p''-ac)]}}{2(p''-ac)}$$

$$z = \frac{-B \mp\sqrt{B^2 - 4p'(p-ab)\,(p''-ac)]}}{2(p-ab)}$$

where $cp - ap' - bp'' + abc = A$, $cp - ap' + bp'' - abc = B$.

19) E. $$\left\{\begin{array}{l} x^2 + y^2 + z^2 = k \\ ax + a'y + a''z = 0 \\ bx + b'y + b''z = 0 \end{array}\right\}$$

S. $x = (a'b'' - a''b')A,\ y = (a''b - ab'')A,$

$$z = (ab' - a'b)A$$

where $\pm\sqrt{\dfrac{k}{(ab'-a'b)^2+(a'b''-a''b')^2+(a''b-ab'')^2}} = A$

20) E. $$\left\{\begin{array}{l} axy + bx + cy + d = 0 \\ a'yz + b'y + c'z + d' = 0 \\ a''zx + b''z + c''x + d'' = 0 \end{array}\right\}$$

S. The elimination of x and y leads to the following quadratic equation.

$$(a''x + b'')\,[(bb' - ad')x + b'd - ad'] +$$
$$(c''x + d'')\,[(ac' - a'b)x + cc' - a'd] = 0$$

and if x is determined by this equation, the values of y and z are easily found.

(The two last kinds of equations will be found generalized in page 184.)

4) SOLUTION OF EQUATIONS OF HIGHER DEGREES.

a. CARDAN'S FORMULA.

E. $x^3 = Px + Q$

S. $$x = \sqrt[3]{\frac{Q + \sqrt{\left(Q^2 - \frac{4P^3}{27}\right)}}{2}} + \sqrt[3]{\frac{Q - \sqrt{\left(Q^2 - \frac{4P^3}{27}\right)}}{2}}$$

EXAMPLES.

1) E. $x^3 - 3x - 2 = 0$

S. $x = 2$

2) E. $x^3 + 12x + 63 = 0$

S. $x = -3$

3) E. $x^3 - 21x + 344 = 0$

S. $x = -8$

4) E. $x^3 - 6x - 40 = 0$

S. $x = \sqrt[3]{(20 + \sqrt{392})} + \sqrt[3]{(20 - 392)} =$

$\sqrt[3]{(2 + \sqrt{2})^3} + \sqrt[3]{(2 - \sqrt{2})^3} = 4$

5) E. $x^3 + 3x - 14 = 0$

S. $x = \sqrt[3]{(7 + \sqrt{50})} + \sqrt[3]{(7 - \sqrt{50})} =$

$\sqrt[3]{(1 + \sqrt{2})^3} + \sqrt[3]{(1 - \sqrt{2})^3} = 2$

6) E. $x^3 - \frac{15}{2}x + 290\frac{1}{2} = 0$

S. $x = \sqrt[3]{\frac{-581 + \sqrt{337311}}{4}} + \sqrt[3]{\frac{-581 - \sqrt{337311}}{4}}$

$= \sqrt[3]{\left(\frac{-7 + \sqrt{39}}{2}\right)^3} + \sqrt[3]{\left(\frac{-7 - \sqrt{39}}{2}\right)^3} = -7$

7) E. $x^3 - 12x^2 + 57x - 94 = 0$

S. $x = 4 + \sqrt[3]{3} - \sqrt[3]{9} = 3{,}36216\ldots$

8) E. $x^3 - 12x - 28 = 0$

S. $x = \sqrt[3]{(14 + \sqrt{132})} + \sqrt[3]{(14 - \sqrt{132})}$

$= 4{,}30213\ldots$

9) E. $x^3 + 6x^2 + 20x + 15 = 0$

$$\text{S. } x = -2 + \sqrt[3]{\frac{81 + \sqrt{12705}}{18}} + \sqrt[3]{\frac{81 - \sqrt{12705}}{18}}$$

$$= -2 + \sqrt[3]{\left(\frac{3 + \sqrt{105}}{6}\right)^3} + \sqrt[3]{\left(\frac{3 - \sqrt{105}}{6}\right)^3} = -1$$

10) E. $x^3 - 15x^2 + 71x - 297 = 0$

$$\text{S. } x = 5 + \sqrt[3]{\frac{864 + \sqrt{746304}}{9}} + \sqrt[3]{\frac{864 - \sqrt{746304}}{9}}$$

$$= 5 + \sqrt[3]{\left(\frac{9 + \sqrt{69}}{3}\right)^3} + \sqrt[3]{\left(\frac{9 - \sqrt{69}}{3}\right)^3} = 11$$

11) E. $x^3 - 12x^2 + 36x - 7 = 0$

$$\text{S. } x = 4 + \sqrt[3]{\frac{-9 + \sqrt{-175}}{2}} + \sqrt[3]{\frac{-9 - \sqrt{-175}}{2}}$$

$$= 4 + \sqrt[3]{\left(\frac{3 + \sqrt{-7}}{2}\right)^3} + \sqrt[3]{\left(\frac{3 - \sqrt{-7}}{2}\right)^3} = 7$$

b. BY FINDING THEIR RATIONAL ROOTS.*

1) E. $x^3 - 9x^2 + 26x - 24 = 0$

R. 2, 3, 4

2) E. $x^3 - 8x^2 + 5x + 14 = 0$

R. -1, 2, 7

---

* The letter R stands for Roots of the equation.

3) E. $x^3 - 49x - 120 = 0$
R. $-3, \ -5, \ 8$

4) E. $x^3 - 18x^2 + 87x - 110 = 0$
R. $2, \ 5, \ 11$

5) E. $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$
R. $1, \ 2, \ 3, \ 4$

6) E. $x^4 - 45x^2 - 40x + 84 = 0$
R. $1, \ -2, \ -6, \ 7$

7) E. $x^4 + 29x^3 + 287x^2 + 1147x + 1560 = 0$
R. $-3, \ -5, \ -8, \ -13$

8) E. $x^3 + \frac{17}{4}x^2 - \frac{79}{8}x + \frac{15}{4} = 0$
R. $\frac{1}{2}, \ \frac{5}{4}, \ -6$

9) E. $x^3 - \frac{13}{12}x^2 + \frac{3}{8}x - \frac{1}{24} = 0$
R. $\frac{1}{2}, \ \frac{1}{3}, \ \frac{1}{4}$

10) E. $x^3 - \frac{14}{3}x^2 + 7x - \frac{10}{3} = 0$
R. $1, \ \frac{5}{3}, \ 2$

11) E. $x^3 + \frac{8}{5}x^2 + \frac{9}{100}x - \frac{9}{100} = 0$
R. $\frac{1}{5}, \ -\frac{3}{10}, \ -\frac{3}{2}$

12) E. $x^3 - \frac{39}{28}x^2 + \frac{27}{56}x - \frac{3}{56} = 0$
R. $\frac{1}{2}, \ \frac{1}{7}, \ \frac{3}{4}$

13) E. $x^3 - \frac{139}{24}x^2 + \frac{329}{48}x + \frac{55}{16} = 0$
R. $-\frac{3}{8}, \ \frac{5}{2}, \ \frac{11}{3}$

14) E. $x^3 + \frac{82}{15}x^2 - \frac{173}{5}x - \frac{126}{5} = 0$
R. $-\frac{2}{3}, \ \frac{21}{5}, \ -9$

15) E. $x^4 - \frac{19}{4}x^3 + \frac{49}{8}x^2 - \frac{11}{4}x + \frac{3}{8} = 0$
R. $\frac{1}{2}, \ \frac{1}{4}, \ 1, \ 3$

16) E. $x^4 - \frac{41}{8}x^3 + \frac{287}{32}x^2 - \frac{393}{64}x + \frac{45}{32} = 0$
R. $\frac{1}{2}$, $\frac{3}{4}$, $\frac{15}{8}$, 2

17) E. $x^3 - 14x^2 - 5x + 70 = 0$
R. 14, $+\sqrt{5}$, $-\sqrt{5}$

18) E. $x^3 - 13x^2 + 49x - 45 = 0$
R. 5, $4 + \sqrt{7}$, $4 - \sqrt{7}$

19) E. $x^3 - 13x^2 + 38x + 16 = 0$
R. 8, $\frac{5}{2} + \frac{1}{2}\sqrt{33}$, $\frac{5}{2} - \frac{1}{2}\sqrt{33}$

20) E. $x^3 - 6x^2 + 19x - 44 = 0$
R. 4, $1 + \sqrt{-10}$, $1 - \sqrt{-10}$

21) E. $x^3 + \frac{7}{6}x^2 + \frac{13}{2}x + \frac{21}{2} = 0$
R. $-\frac{3}{2}$, $\frac{1}{6} + \frac{1}{6}\sqrt{-251}$, $\frac{1}{6} - \frac{1}{6}\sqrt{-251}$

22) E. $x^4 + x^3 - 24x^2 + 43x - 21 = 0$
R. 1, 3, $-\frac{5}{2} + \frac{1}{2}\sqrt{53}$, $-\frac{5}{2} - \frac{1}{2}\sqrt{53}$

23) E. $x^5 - 3x^4 - 8x^3 + 24x^2 - 9x + 27 = 0$
R. 3, 3, -3, $+\sqrt{-1}$, $-\sqrt{-1}$

24) E. $x^5 - \frac{3}{2}x^4 - 6x^3 + 9x^2 - 13x + 19\frac{1}{2} = 0$
R. $\frac{3}{2}$, $+\sqrt{(3 + \sqrt{22})}$, $-\sqrt{(3 + \sqrt{22})}$,
$+\sqrt{(3 - \sqrt{22})}$, $-\sqrt{(3 - \sqrt{22})}$

5) TWO GENERAL CASES OF EQUATIONS, WITH SEVERAL UNKNOWN QUANTITIES, WHICH ADMIT OF SOLUTION.

I. Let x', x'', x''' . . . . . $x^{n'}$, denote n unknown quantities, and let us have an equation of the form

$$\begin{gathered} a'x'^m + a''x''^m + a'''x'''^m + a''''x''''^m + \ldots\ldots \\ + a^{n'}(x^{n'})^m = K \end{gathered}$$

and besides $(n - 1)$ equations of the following form, wherein the unknown quantities do not exceed the first degree.

$$\begin{gathered} b'x' + b''x'' + b'''x''' + b''''x'''' + \ldots\ldots + b^{n'}x^{n'} = 0 \\ c'x' + c''x'' + c'''x''' + c''''x'''' + \ldots\ldots + c^{n'}x^{n'} = 0 \\ d'x' + d''x'' + d'''x''' + d''''x'''' + \ldots\ldots + d^{n'}x^{n'} = 0 \\ \&c. \end{gathered}$$

Then, by the solution of these last equations, we can express all the quantities sought, in terms of one of them thus: $x'' = A''x'$, $x''' = A'''x'$, $x'''' = A''''x'$ .... $x^{n'} = A^{n'}x'$, where A'', A''', A'''', ... A^{n}, represent known quantities; which, being substituted in the first equation, give

$$x' = \sqrt[m]{\frac{K}{a' + a''A''^m + a'''A'''^m + a''''A''''^m \ldots + a^{n'}(A^{n'})^m}}$$

whence are obtained the values of x'', x''' ..... $x^{n'}$. This may be still further generalised, but is left to the sagacity of the student.

II. Let there be the n following equations between the n unknown quantities x', x'', x''' .... $x^{n'}$.

$$\begin{array}{lllll} a'x'x'' & + b'x' & + c'x'' & + d' & = 0 \\ a''x''x''' & + b''x'' & + c''x''' & + d'' & = 0 \\ a'''x'''x'''' & + b'''x''' & + c'''x'''' & + d''' & = 0 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots \\ a^{(n-1)'}x^{(n-1)'}x^{n'} & + b^{(n-1)'}x^{(n-1)'} & + c^{(n-1)'}x^{n'} & + d^{(n-1)'} & = 0 \\ a^{n'}x^{n'}x' & + b^{n'}x^{n'} & + c^{n'}x' & + d^{n} & = 0 \end{array}$$

From the first equation find the value of x'' in terms of x', which substitute in the 2d, will give you the value of x''' in terms of x'; and so on throughout, until at length $x^{n'}$ is expressed in terms of x', in the following form:

$$x^{n'} = \frac{Ax' + B}{Cx' + D}.$$

If this value be substituted in the last equation, there results a quadratic in x'; whereby x' may be found, and thereby all the other unknown quantities.

. . . .

If the roots of an equation be given, the equation itself may be found. How? What is the equation, for example, whose roots are 1, 3, -1, -4? What is that whose roots are 6, $2+3\sqrt{-1}$, $2-3\sqrt{-1}$? The coefficients of an equation bear certain relations to their roots; which are they?

When the three equations

$$\left\{\begin{array}{r} x + y + z = a \\ xy + xz + yz = b \\ xyz = c \end{array}\right\}$$

or when

$$\left\{\begin{array}{r} x + y + z + w = a \\ xy + xz + xw + yz + yw + zw = b \\ xyz + xyw + xzw + yzw = c \\ xyzw = d \end{array}\right\}$$

are given, then we can obtain, in the first case, an equation of the third, and in the second, an equation of the fourth degree, which gives all the unknown quantities at once. How is this equation formed?—When m roots of an equation of the nth degree are known, then the determination of the n remaining ones requires but the solution of an equation of the $(n - m)$th degree. How is this equation obtained? When the degree of an equation is an odd number, it has at least one real root. Why? When $h + k\sqrt{-1}$ is any imaginary root of an equation, $h - k\sqrt{-1}$ must also be a root of the same equation. How can this be proved? If it be granted, (and indeed, it can be rigorously demonstrated) that all imaginary quantities can be represented by the formula $h + k\sqrt{-1}$, how many imaginary, and how many real roots can an equation of the nth degree have, when n is even, and when n is odd? When Cardan's formula gives an imaginary result, has the equation therefore no real root; or does this merely show, that the form, which we wished the root to assume, was impossible?

XIV. SOLUTION OF EQUATIONS BY APPROXIMATION.

1) EQUATIONS WITH ONE UNKNOWN QUANTITY.

FIRST METHOD.

Let $X = 0$, represent any equation involving the unknown quantity x. Let w be a value of x, found by trial, and differing from the real value by something less than unity. Now, if $w + h$ is substituted for x in that equation, it is evident that $h < 1$; consequently, the higher powers of h may be omitted in the result, by which means we shall obtain, instead of the equation $X = 0$, one of the form $A + Bh = 0$; where A, B, are known quantities. From this equation we may obtain h, consequently also $x\,(= w + h)$, at least by approximation. This new value of x may then be treated as the first one w, and by repeating the same operation, the value of x may be found to any degree of approximation.

By the application of this principle to the general equation

$$x^m + ax^{m-1} + bx^{m-2} + \ldots\ldots\ldots\ldots\, hx^2 + hx + l = 0$$

we obtain the following expression for each successive approximate value:

$$\frac{(m-1)w^m + (m-2)aw^{m-1} + (m-3)bw^{m-2} + \ldots + 1hw^2 - l}{mw^{m-1} + (m-1)aw^{m-2} + (m-2)bw^{m-3} + \ldots + 2hw + k}$$

consequently, for an equation of the third degree, $x^3 + ax^2 + bx + c = 0$, we have

$$x = \frac{2w^3 + aw^2 - c}{3w^2 + 2aw + b}$$

For the fourth degree, $x^4 + ax^3 + bx^2 + cx + d, = 0$, we get

$$x = \frac{3w^4 + 2aw^3 + bw^2 - d}{4w^3 + 3aw^2 + 2bw + c}$$

For the fifth degree, $x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0$, we find

$$x = \frac{4w^5 + 3aw^4 + 2bw^3 + cw^2 - e}{5w^4 + 4aw^3 + 3bw^2 + 2cw + d}$$

&c.

EXAMPLES.

1) E. $x^3 = 2$
$R = 1, R' + \frac{4}{3}, R'' = \frac{91}{72}, R''' = \frac{2253638}{1788696}$, &c.*

2) E. $x^3 = 30$
$R = 3, R' = \frac{28}{9}, R'' = \frac{65774}{21168}$, &c.

3) E. $x^3 - 12x^2 + 57x - 94 = 0$
$R = 3, R' = \frac{10}{3}, R'' = \frac{938}{279}$, &c.

4) E. $x^3 - 15x^2 + 72x - 109 = 0$
$R = 5, R' = \frac{16}{3}, R'' = \frac{385}{72}$, &c.

5) E. $x^3 - 13x^2 + 38x + 17 = 0$
$R = 8, R' = \frac{175}{22}, R'' = \frac{1778984}{223674}$, &c.

6) E. $x^3 + 2x^2 + 3x - 52 = 0$
$R = 3, R' = \frac{62}{21}, R'' = \frac{1119676}{379323}$, &c.

7) E. $x^3 - 12x + 132 = 0$
$R = 6, R' = \frac{47}{8}, R'' = \frac{137615}{23436}$, &c.

8) E. $x^4 - 4x^3 + 18 = 0$
$R = 2, R' = \frac{17}{8}, R'' = \frac{137597}{64736}$, &c.

9) E. $x^4 + 8x^2 + 16x - 440 = 0$
$R = 4, R' = \frac{167}{42}, R'' = \frac{4086104771}{1030204056}$, &c.

To carry this approximation any further, it is necessary to change the vulgar fraction we found for x'', into a decimal fraction; but then it would be needless to develope the fraction to more than 3 decimal places; because it is a rare case, that the third approximation gives the root exact to more than three places.

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* R, R′, R″, &c, denote the successive approximate values of x. There is no need of calculating more than one root at a time. The same process will apply to the other possible roots of the equation.

SECOND METHOD.

Let $X = 0$, be an equation in x. Its root may be found by continued fractions, according to the general principle laid down in VIII, page 142, we proceed as follows:

Put $a + \frac{1}{x^I}$ instead of x, and by that means change the equation $X = 0$ into $X^I = 0$, involving x^I. Then let a^I and $a^I + 1$, be the two whole numbers between which the true value of the root lies; and put $a^I + \frac{1}{x^{II}}$ instead of x^I, and change thereby the equation $X^I = 0$, into another $X^{II} = 0$, involving x^{II}. Then again, if a root of this last equation falls between a^{II} and $a^{II} + 1$, put $a^{II} + \frac{1}{x^{III}}$ for x^{II}, and so on; and you will obtain the root x of the given equation expressed by the following continued fraction:

$$x = a + \frac{1}{x^I} = a + \cfrac{1}{a^I + \cfrac{1}{x^{II}}} = a + \cfrac{1}{a^I + \cfrac{1}{a^{II} + \&c.}}$$

which, when reduced, will give the approximating value of the root x, of the given equation, which is sought.

EXAMPLES.

$$1)\ x^3 - 2 = 0$$

. . . .

$$\begin{array}{l|l}
X\ = x^3 - 2 = 0 & x = 1 + \frac{1}{x^I} \\
X^I\ = x^{I3} - 3x^{I2} - 3x^I - 1 = 0 & x^I = 3 + \frac{1}{x^{II}} \\
X^{II} = 10x^{II3} - 6x^{II2} - 6x^{II} - 1 = 0 & x^{II} = 1 + \frac{1}{x^{III}} \\
X^{III} = 3x^{III3} - 12x^{III2} - 24x^{III} - 10 = 0 & x^{III} = 5 + \frac{1}{x^{IV}} \\
X^{IV} = 55x^{IV3} - 81x^{IV2} - 33x^{IV} - 3 = 0 & x^{IV} = 1 + \frac{1}{x^V} \\
X^V\ = 62x^{V3} + 30x^{V2} - 84x^V - 55 = 0 & x^V = 1 + \frac{1}{x^{VI}} \\
\qquad\qquad\&c. &
\end{array}$$

Approximating Values, 1, $\frac{4}{3}$, $\frac{5}{4}$, $\frac{29}{23}$, $\frac{34}{27}$, $\frac{63}{50}$, &c.
True Root, 1,25992 . . . . .

$$2)\ x^3 - 15x^2 + 63x - 50 = 0$$

. . . .

$$\begin{array}{l|l}
X = x^3 - 15x^2 + 63x - 50 = 0 & x = 1 + \frac{1}{x^I} \\
X^I = x^{I3} - 36x^{I2} - 12x^I - 1 = 0 & x^I = 35 + \frac{1}{x^{II}} \\
X^{II} = 806x^{II3} - 1167x^{II2} - 69x^{II} - 1 = 0 & x^{II} = 1 + \frac{1}{x^{III}} \\
X^{III} = 431x^{III3} - 15x^{III2} - 1251x^{III} - 806 = 0 & x^{III} = 1 + \frac{1}{x^{IV}} \\
\qquad\qquad\&c. &
\end{array}$$

Approximating Values, 1, $\frac{36}{33}$, $\frac{37}{36}$, $\frac{73}{71}$, &c.
True Root, 1,02803 . . . . . .

3) $x^3 - 12x^2 + 45x - 53 = 0$

. . . .

$X = x^3 - 12x^2 + 45x - 53 = 0$	$x = 5 + \frac{1}{x^I}$
$X^I = 3x^{I3} - 3x^I - 1 = 0$	$x^I = 1 + \frac{1}{x^{II}}$
$X^{II} = x^{II3} - 6x^{II2} - 9x^{II} - 3 = 0$	$x^{II} = 7 + \frac{1}{x^{III}}$
$X^{III} = 17x^{III3} - 54x^{III2} - 15x^{III} - 1 = 0$	$x^{III} = 3 + \frac{1}{x^{IV}}$
$X^{IV} = 73x^{V3} - 120x^{IV2} - 99x^{IV} - 17 = 0$	$x^{IV} = 2 + \frac{1}{x^V}$
$X^V = 111x^{V3} - 297x^{V2} - 318x^V - 73 = 0$	$x^V = 3 + \frac{1}{x^{VI}}$
$X^{VI} = 703x^{VI3} - 579x^{VI2} - 702x^{VI} - 111 = 0$	$x^{VI} = 1 + \frac{1}{x^{VII}}$
&c.	

Approximating Values, 5, 6, $\frac{47}{8}$, $\frac{147}{25}$, $\frac{341}{98}$, $\frac{1170}{199}$, $\frac{1511}{257}$, &c.
True Root, 5,879385 . . . . .

4) $x^3 - 12x^2 + 57x - 94 = 0$

. . . .

$X = x^3 - 12x^2 + 57x - 94 = 0$	$x = 3 + \frac{1}{x^I}$
$X^I = 4x^{I3} - 12x^{I2} + 3x^I - 1 = 0$	$x^I = 3 - \frac{1}{x^{II}}$
$X^{II} = 8x^{II3} - 39x^{II2} + 24x^{II} - 4 = 0$	$x^{II} = 4 + \frac{1}{x^{III}}$
$X^{III} = 20x^{III3} - 96x^{III2} - 57x^{III} - 8 = 0$	$x^{III} = 5 + \frac{1}{x^{IV}}$
$X^{IV} = 193x^{IV3} - 483x^{IV3} - 204x^{IV} - 20 = 0$	$x^{IV} = 3 - \frac{1}{x^V}$
&c.	

Approximating Values, 3, $\frac{10}{3}$, $\frac{37}{11}$, $\frac{195}{58}$, $\frac{622}{185}$, &c.
True Root, 3,36216 . . . . . .

$$5)\ x^3 - 12x - 28 = 0$$

. . . .

$$\begin{array}{l|l}
X = x^3 - 12x - 28 = 0 & x = 4 + \frac{1}{x^I} \\
X^I = 12x^{I3} - 36x^{I2} - 12x^I - 1 = 0 & x^I = 3 + \frac{1}{x^{II}} \\
X^{II} = 37x^{II3} - 96x^{II2} - 72x^{II} - 12 = 0 & x^{II} = 3 + \frac{1}{x^{III}} \\
X^{III} = 93x^{III3} - 351x^{III2} - 237x^{III} - 37 = 0 & x^{III} = 4 + \frac{1}{x^{IV}} \\
X^{IV} = 649x^{IV3} - 1419x^{IV2} - 765x^{IV} - 93 = 0 & x^{IV} = 3 + \frac{1}{x^V} \\
\text{\&c.} &
\end{array}$$

Approximating Values, 4, $\frac{13}{3}$, $\frac{43}{10}$, $\frac{185}{43}$, $\frac{598}{139}$, &c.
True Root, 4,30213 . . . . .

In this method several abbreviations and reductions may be made, which however do not belong hither. It may also be advantageously combined with the first method, when the approximation is somewhat advanced.

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2) EQUATIONS WITH TWO UNKNOWN QUANTITIES.

Let $X = 0$, $X_I = 0$, be two equations, involving x and y, and let it be assumed, that we know already pretty near values of x and y; required to find nearer approximations to them.

I. Let $x = a$, $y = b$, be these values. Put $a + h$ for x, and $b+k$ for y, in both equations $X=0$, $X_{I}=0$, retaining in the result neither the *products* of h by k, nor the *powers* higher than the first. Then the equation $X=0$, $X_{I}=0$, will be changed into

$$\begin{aligned} A+Bh+Ck&=0\\ A'+B'h+C'k&=0 \end{aligned}$$

where A, B, C, A', B', C', are known quantities. If from these h and k are determined, we get nearer approximations to x and y, namely, $x=a+h$, $y=b+k$.

II. We proceed with these values exactly as we before did with a, b, and, by that means, obtain two still more approximate values of x and y.

III. This process being repeated, will give the approximations to any required degree of accuracy. Moreover, in these calculations logarithms may be used with advantage.

EXAMPLES.

Let the given equations be

$$\begin{aligned} x^7-5x^2y^4+1506&=0\\ y^5-3x^4y-103&=0 \end{aligned}$$

the values, $x=2$, $y=3$, nearly satisfy them.

FIRST CORRECTION.

$$\begin{aligned} 14-1172h-2160k&=0\\ -4-288h+357k&=0 \end{aligned}$$

Whence, $h=-0{,}0035$, $k=+0{,}0084$

Therefore, $x=1{,}9965$, $y=3{,}0084$

SECOND CORRECTION.

$$\begin{aligned} -0{,}486-1189{,}170h-2170{,}576k&=0\\ 0{,}026-287{,}293h+361{,}890k&=0 \end{aligned}$$

Whence, $h=-0{,}000113$, $k=-0{,}000161$

Therefore, $x=1{,}996387$, $y=3{,}008239$

The last approximations to x and y are right to five decimal places. Thus, if no greater degree of exactness is required, no further approximations are needed.

. . . .

This method admits of being generalized thus. Let there be n equations $X = 0$, $X^{I} = 0$, $X^{II} = 0$, &c, involving the n unknown quantities x, y, z, &c. Then if a, b, c, are values of x, y, z, &c, differing from the true ones by less than unity, substitute instead of x, y, z, &c, $a + h$, $b + k$, $c + l$, &c, and omit in the results the products and higher powers of the quantities h, k, l, &c, and you will obtain n equations between these quantities, of the form

$$A + Bh + Ck + Dl + \ldots \text{ \&c,} = 0$$

from which may be found the quantities h, k, l, &c, and thereby also the nearer approximations to x, y, z, &c.— With these new values proceed as before with a, b, c, &c, until the values of x, y, z, &c, are sufficiently near determined.

The spirit of this method, which consists in *successive corrections*, is the foundation of almost every method of approximation. It merits, on account of its simplicity, and manifold application, to find a place in every treatise on Algebra; although the generalization of it, together with the reductions of which it admits, belongs to the differential calculus.

SECTION III.

CONTAINING

PROBLEMS FOR APPLYING AND PRACTISING THE PRECEDING RULES.

XV. PROBLEMS FOR THE EQUATIONS OF THE FIRST DEGREE, WITH ONE UNKNOWN QUANTITY.

1) Two capitalists calculate their fortunes, and it appears that one is twice as rich as the other, and that together they possess \$38700. What is the capital of each?

Ans. The one has \$12900, the other \$25800.

2) The sum of \$2500 is to be divided between two brothers, so that one receives \$4 as often as the other receives one. How much does each receive?

Ans. The one receives \$500, the other \$2000.

3) A person has \$2640, among which $4\frac{1}{2}$ times as many notes as species. How much has he of each sort?

Ans. \$2160 in notes, and \$480 in species.

4) Divide the number 237 into two such parts, that the one may be contained in the other $1\frac{1}{4}$ times. What are these parts?

Ans. $105\frac{1}{3}$, and $131\frac{2}{3}$.

What have these four problems in common?

5) To find two such numbers, that the one may be m times as great as the other, and that their sum $= a$. What are these two numbers?

Ans. $\frac{a}{m+1}$ and $\frac{ma}{m+1}$.

6) The sum of $1200 is to be divided between two persons, A and B, that A's share is to B's share as 2 to 7. How much does each receive?

Ans. A 266\frac{2}{3}$, B 933\frac{1}{3}$.

How may this problem be generalized?

7) Divide a number a into two such parts, that the first part is to the second as m to n. What are the parts?

Ans. $\frac{ma}{m+n}$, $\frac{na}{m+n}$.

What has this problem in common with the 5th? And how may the one be reduced to the other?

8) How much money have I in my pocket when the 4th and 5th part of it amount together to $2,25?

Ans. $5.

9) Two friends met a horse-dealer leading a horse, which they resolved to buy jointly. When they had agreed as to the price, they found that the one was able to pay only the 5th, and the other the 7th part; this then they put together and paid the seller therewith, on account, $48. What was the price of the horse?

Ans. $140.

This and the preceding problem are similar. How may this and all similar problems be expressed in general terms?

10) Find a number such, that when it is divided by m and by n, and the quotients afterwards added, the sum $= a$.

Ans. $\frac{mna}{m+n}$.

11) Divide the number 46 into two unequal parts, so that when the one is divided by 7, and the other by 3, the quotients together may amount to 10. What are these parts?

Ans. 28 and 18.

12) Divide a number a into two such parts, that the sum of the quotients which are obtained when the one part is divided by m, and the other by n, may be equal to the number b. What are these parts?

Ans. $\frac{m(nb-a)}{n-m}$, $\frac{n(mb-a)}{m-n}$.

13) In a company of 266 persons, consisting of officers merchants and students, there were 4 times as many merchants, and twice as many officers as students. How many were there of each class?

Ans. 38 students, 152 merchants, and 76 officers.

14) A fortress has a garrison of 2600 men; among whom there are 9 times as many foot soldiers, and three times as many artillery soldiers as cavalry. How many are there of each corps?

Ans. 200 cavalry, 600 artillery, and 1800 foot.

15) All my journeyings taken together, says a traveller, amount to 3040 miles; of which I have travelled $3\frac{1}{2}$ times as much by water as on horseback, and $2\frac{1}{4}$ times as much on

foot as by water. How many miles did this man travel by each of these three ways?

Ans. 240 miles on horseback, 840 miles by water, and 1960 miles on foot.

What have the problems 13, 14, 15, common?

16) Divide the number a into three such parts, that the 2d may be m times, and the 3d n times as great as the first. What are these parts?

Ans. $\frac{a}{1+m+n}$, $\frac{ma}{1+m+n}$, $\frac{na}{1+m+n}$.

17) When I multiply a certain number by 4, and divide the product by 3, I obtain 24. What is that number?

Ans. 18.

18) A field of 864 square rods is to be divided among three farmers, A, B, C, so that A's part shall be to B's as 5 to 11, and C may receive as much as A and B together. How much does each receive?

Ans. A 135, B 297, C 432 square rods.

19) Divide $1170 among three persons, A, B, C, proportionally to their ages. Now B is one third part older than A, and C twice as old as A. How much does each receive?

Ans. A $270, B $360, C $540.

20) On an approaching war three towns, A, B, C, are to furnish their complements of 594 men; the division is to be made in proportion to their population. Now the population of A is to that of B as 3 is to 5; whilst the population of B is to that of C as 8 is to 7. How many men must each town furnish?

Ans. A 144, B 240, C 210 men.

21) A bankrupt leaves $21000 to be divided among four creditors, A, B, C, D, in proportion to their claims. Now A's claim is to B's as 2 to 3; B's claim to C's as 4 to 5, and C's claim to D's as 6 to 7. How much then does each creditor receive?

Ans. A $3200, B $4800, C $6000, D $7000.

22) Divide the number a into three such parts, that the first shall be to the 2d as m to n, and the 2d part to the 3d as $p : q$. What are these parts?

Ans. $$\frac{mpa}{mp + np + nq}, \quad \frac{npa}{mp + np + nq}, \quad \frac{nqa}{mp + np + nq}.$$

23) The 3d part of my income, says a person, I expend in board and lodging, the 8th part in clothes and washing, the 10th part in incidental expenses, and yet I save $318 yearly. What is the amount of his yearly income?

Ans. $720.

24) A merchant finds, by a fortunate speculation with his floating capital, he has gained 15 per cent, and that by this means it has increased to $15571. What was his capital?

Ans. $13540.

25) A capital was put out for one year at 4½ per cent per annum; at the expiration of the year there was received back $13167, capital and interest. How much did the capital amount to?

Ans. $12600.

26, The rent of an estate is this year by 8 per cent greater than it was the last year. This year's rent is $1890. What was the rent of last year?

Ans. $1750.

27) A pound of a certain commodity is sold for 45 cents; out of this there accrues to the seller a profit of 12½ per cent. How much did it cost per cwt?

Ans. $44,80.

28) What capital is it, which with 5 years' interest, (the interest calculated at 4 per cent), amounts to $8208?

Ans. $6840.

29) A gamester lost in the first game the 6th, and in the second the 10th part of the money he had about him, but won in the third game the third part of it back. He counted his money, and found he had won $3. How much had he about him when he began to play?

Ans. $45.

30) There are two numbers whose sum is 96, and of which one is greater than the other by 16. What are these numbers?

Ans. 40, and 56.

31) Two merchants having left off business, their profits, amounting to $1200, are to be divided between them in such a way, that one partner receives only half as much as the other, exclusive, however, of $50 for his labor. How much does each receive?

Ans. The one $766⅔, the other $433⅓.

32) Divide $1520 among three persons, A, B, C, so that B may receive $100 more than A; and C $270 more than B. How much does each receive?

Ans. A $350, B $450, C $720.

33) A widow, according to the will of her deceased husband, shall divide a sum of $7500 with her two sons and three daughters, so that each son may receive twice as much as each daughter; she herself, however, is to receive $50 more than all her children together. What was her share? And what was the share of each of her children?

Ans. The widow's share is $4000, each son receives $1000, and each daughter $500.

34) A company of 90 persons consists of men, women and children. The men are 4 in number more than the women, the children 10 more than adults. How many men, women, and children, are there in the company?

Ans. 22 men, 18 women, 50 children.

35) A forest of 8000 square feet is to be divided among three farm-houses, A, B, C, that B may receive 276 square feet less than A, but C 1112 square feet more than B. How many square feet will each receive?

Ans. A 2480, B 2204, C 3316.

36) A father gives to his five sons $1000, which they are to divide according to their ages, so that each elder son shall receive $20 more than his next younger brother. What is the share of the youngest?

Ans. $160.

37) A certain sum of money is to be divided amongst three persons, A, B, C, as follows: A shall receive $3000

less than the half of it, B $1000 less than the third part, and C is to receive $800 *more* than the fourth part of the whole sum. What is the sum to be divided? and what does each receive?

Ans. The whole sum is $38400. A receives $16200, B $11800, C $10400.

38) A dying man bequeaths in his will the half of his property to his wife, one sixth part to each of his two sons, the 12th part to his faithful servant, and the remaining $600 to the poor. What was the amount of the property bequeathed?

Ans. $7200.

39) A meadow of 2850 square feet is to be divided amongst three landlords, A, B, C; A's share shall be to B's as 6 to 11, and C is to have 300 square feet more than A and B together. What is each one's share?

Ans. A's 450, B's 825, C's 1575 square feet.

40) A father leaves behind him four sons, A, B, C, D, and a fortune of $2520, which they are to divide amongst them as follows: C is to receive $360, B as much as C and D together; but A twice as much as B, less $1000. How much does A, B, and D receive?

Ans. A $760, B $880, D $520.

41) Five heirs are to divide an inheritance of $5600; B shall receive twice as much as A, and $200; C three times as much as A, less $400; D the half of what B and C receive together, and $150 more; but D the fourth part of what his three participators receive, together with $475. How much does each receive?

Ans. A $500, B $1200, C $1100, D $1300, E $1500.

42) Five gamesters have lost jointly \$$40\frac{5}{8}$; B's loss amounts to $\frac{1}{2}$ dollar more than triple A's; C's loss is \$2 less than twice B's; D lost $\frac{1}{4}$ dollar less than A and B together; and E twice as much as B, less $\frac{1}{8}$ dollar. How much did each of them lose?

Ans. A \$2, B \$$6\frac{1}{2}$, C \$11, D \$$8\frac{1}{4}$, E \$$12\frac{7}{8}$.

43) A certain part of a commodity which weighed 40 cwt. was sold, and there remained 8 cwt. more than was sold. How many cwt. were sold?

Ans. 16.

44) I once had \$42 in my possession; of this I gave away a certain sum, and there remained still three times as much as I had given away. How much did I give away?

Ans. \$$10\frac{1}{2}$.

45) Two gentlemen, A and B, play at billiards; A, before he began to play, had \$42, and B \$24. After a match, in which each lost and won in turns, A found himself in possession of five times as much as B had remaining. How much did A win?

Ans. \$13.

46) The garrison of a certain town consists of 1250 men, partly infantry, and partly cavalry. Each horse soldier receives \$5, and each foot soldier \$3. Now, if the monthly pay of the garrison amounts to \$4150, how many horse, and how many foot soldiers did it contain?

Ans. 200 cavalry, 1050 infantry.

47) A mason, 12 journeymen, and 4 assistants, receive together \$72 wages for a certain time. The mason receives \$1 daily, each journeyman $\frac{1}{2}$ dollar, and each assistant 25

cents. How many days must they have worked for this money?

Ans. 9 days.

48) A capitalist derives from the sums he has out at interest, a yearly income of $2940; four fifths of it bears 4 per cent, and one fifth 5 per cent. How much money has he out at interest?

Ans. $70000.

49) I have a certain number in my thoughts, says A to B, try to guess it. I multiply it by 7, add 3 to the product, divide this by 2, subtract 4 from the quotient, and obtain 15. Now, what number is it?

Ans. 5.

50) Find three such numbers, that the 2d divided by the first gives 2 for a quotient and 1 for a remainder; the 3d divided by the 2d, 3 for the quotient, with the remainder 3; and the sum of these three numbers shall be 70. What numbers are they?

Ans. 7, 15, and 48.

51) How much money have you about you, said a person to a friend. I have as many dollars about me, said the other, that if I multiply their number by 5, subtract 3 from the product, multiply, again, the remainder by 4, add 2 to the product, then take away from this number the cipher to the right, I obtain $23. How many dollars had he about him?

Ans. $12.

52) An arithmetician desires his scholars to find a number, which he has in his mind, from the following data. If, says

he, you multiply the number by 5, subtract 24 from the product, divide the remainder by 6, and add 13 to the quotient, you will obtain this same number. What number, then, is it?

Ans. 54.

53) A courier who had started from a certain place 10 days ago, is pursued by another from the same place, and by the same way. The first goes 4 miles every day, the other 9. How many days will the second need to overtake the first?

Ans, 8 days.

54) A courier left this place n days ago, and makes a miles daily. He is pursued by another making b miles daily. How many days will the second require to overtake the first?

Ans. $\frac{n\,a}{b - a}$ days.

55) But, in what time will the second courier overtake the first, when it is supposed that the second starts 12 days later than the first, and his speed is to that of the first as 8 is to 3?

Ans. $7\frac{1}{5}$ days.

56) Two bodies start from the same place, one after the other, in a straight line; the 2d starts n seconds later than the first, and its speed is to that of the first as q is to p. In what time will these two bodies be together?

Ans. $\frac{pn}{q - p}$ seconds after the setting out of the 2d.

57) A courier, who goes $31\frac{1}{2}$ miles every 5 hours, is sent from a certain place; when he had been gone 8 hours another was sent after him; and this one, in order to overtake

the first, goes $22\frac{1}{2}$ miles every 3 hours. When will the 2d overtake the first?

Ans. 42 hours after the departure of the 2d courier.

58) When all the conditions in the preceding problem remain the same, except that the first courier, besides the advantage of starting earlier, has this one also, that he started from a place 36 miles farther on the road; in this case, in how many hours will they come together?

Ans. In 72 hours after the departure of the 2d courier.

59) In order to give the preceding problem the requisite generality, let the place from which the first starts be situated a miles in advance on the road; farther, let the number of hours by which he had the start of the other be $= b$; let the speed of the first courier be such, that he goes c miles in d hours, and the speed of the 2d be so great, that he goes e miles in f hours. In how many hours after the departure of the second courier will they be together?

Ans. In $\frac{(ad + bc)f}{de - cf}$ hours.

60) But, in how many hours will they come together, when the first courier, instead of starting from a place a miles in advance, starts from one as many miles backwards?

Ans. In $\frac{(bc - ad)f}{de - cf}$ hours.

What must be done to adopt the solution of problem 59 to this case?

61) A regiment marches from the place A, in a direct line to the place B, and goes $3\frac{1}{2}$ leagues each day; 8 days

after, another regiment marches from the place B, in a straight line to A, and goes $5\frac{1}{6}$ leagues each day. If both places be 80 leagues distant from one another, on what day after the departure of the first will these two regiments meet?

Ans. On the 14th day.

What values must we assign to the quantities a, b, c, d, e, f, in the 59th problem, in order to adapt the solution, there given, to the present case?

62) Two bodies move in opposite directions; one runs c feet in each second, the other C feet. The two places from which they start at the same time, are distant d feet from one another. When will they meet?

Ans. In $\frac{d}{C+c}$ seconds.

63) But, in what time will these two bodies come together, when that which goes C feet each second, runs after the other?

Ans. In $\frac{d}{C-c}$ seconds.

Is the problem, as here stated, always possible? And what is required for it to be possible? What does the expression $\frac{d}{C-c}$ signify, when $C=c$? What does it denote when $C<c$?

64) A hostile corps has set out two days ago from a certain place, and goes 27 miles daily. Another corps wishes to march in pursuit of it from the same place, and so quickly that it may reach the other in 6 days. How many miles must it march daily, to accomplish it?

Ans. 36 miles.

65) Two bodies move after one another in the same direction, the first has a start of d units of measure, (for instance feet), and of t units of time, (for instance seconds); the first in each unit of time runs over c units of measure. How many units of time will the 2d require to overtake the first?

Ans. $\frac{d + ct}{C - c}$ units of time.

66) But how many units of time will the 2d need, when the bodies, instead of running after one another, run towards one another, and all the other conditions remain the same?

Ans. $\frac{d - ct}{C + c}$ units of time.

How may this expression be derived from that found in the preceding problem?

67) At 12 o'clock, both hands of a clock are together. When, and how often will these hands be together in the next 12 hours?

Ans. The hands will meet 11 times; these rencontres will be at $5\frac{5}{11}$ minutes past 1, $10\frac{10}{11}$ minutes after 2, $16\frac{4}{11}$ after 3, and so on, in each successive hour $5\frac{5}{11}$ minutes later.

68) Two bodies move after one another in the circumference of a circle which measures p feet. At first they are distant from each other by an arc measuring d feet; the first moves c feet, the second C feet in a second. When will these two bodies meet for the first time, 2d time, and so on, supposing that they do not disturb each other's motion?

Ans. In $\frac{d}{C - c}$, $\frac{p + d}{C - c}$, $\frac{2p + d}{C - c}$, &c, seconds.

69) But when will they meet, when the first begins to move t seconds sooner than the 2d?

Ans. In $\frac{d+ct}{C-c}$, $\frac{p+d+ct}{C-c}$, $\frac{2p+d+ct}{C-c}$, &c, seconds.

70) But if the first starts t seconds later than the 2d, when will they meet?

Ans. In $\frac{d-ct}{C-c}$, $\frac{p+d-ct}{C-c}$, $\frac{2p+d-ct}{C-c}$, &c, seconds.

71) But if the first, instead of preceding the 2d, runs against it, and starts from the same place, t seconds sooner, when do they meet?

Ans. In $\frac{d-ct}{C+c}$, $\frac{p+d-ct}{C+c}$, $\frac{2p+d-ct}{C+c}$, &c, seconds.

72) But when do they meet, again, when the first starts from the same place as the second does, but with this difference, that it begins to move t seconds later?

Ans. In $\frac{d+ct}{C+c}$, $\frac{p+d+ct}{C+c}$, $\frac{2p+d+ct}{C+c}$, &c, seconds.

Whence arises the difference in the signs in the solutions of these five similar problems, since otherwise the expressions are in all respects alike? Can they not be explained from the opposition which exists between them? What do the expressions in the 68th, 69th and 70th problems mean, when $C=c$? What do they indicate when $C<c$?

* * *

What are compound relations? and where are examples of them found? The relation between two magnitudes may consist of two, three, four, &c, simple ratios. These compound relations are of the greatest importance, both in the pure mathematics, and in their application to the common purposes of life.

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73) From two different sized orifices of a reservoir, the water runs with unequal velocities. We know that the orifices are in size as 5 : 13, and the velocities of the fluid are as 8 : 7; we know farther, that in a certain time there issued from the one, 561 cubic feet more than there did from the other. How much water, then, did each orifice discharge in this space of time?

Ans. The one 440, the other 1001 cubic feet.

74) A dog pursues a hare. Before the dog started the hare had made 50 paces, and this is the distance between them at first. The hare takes 6 paces to the dog's 5; and 9 of the hare's paces are equal to 7 of the dog's. How many paces can the hare take before the dog overtakes her?

Ans. 700 paces.

75) Two bombardiers throw different kinds of shells from a battery. The first had already thrown 36 times, before the 2d began to throw, and throws 8 times to the 2d's 7. On the other hand, the 2d uses as much powder for three of his throws, as the first does for four. How many throws must the 2d make in order to consume as much powder as the first?

Ans. 189 throws.

76) How happens it, says one pedestrian to another, that you are 3000 feet ahead of me, although I took twice as long

steps as you? That I allow, the other answered; but I, on the other hand, took 5 times as many steps as you. If, then, all this is correct, how many feet must each have travelled?

Ans. The first 2000, the 2d 5000 feet.

77) In order to make the foregoing problem more general, let the number of feet by which the 2d pedestrian has outrun the 1st $= a$; let the length of their steps be as $b : c$, and the number of steps as $d : e$. What expressions give the distances they went over?

$$\textit{Ans.}\ \frac{abd}{ce - bd},\ \frac{ace}{ce - bd}.$$

78) Two magnitudes whose difference $= d$, for some reason, are to one another as $m : n$; but for some other reason, respecting which it is assumed that it does not hinder the operation of the other, they are as $m' : n'$. How are these quantities expressed?

Ans. $\frac{mm'd}{nn' - mm'}$, $\frac{nn'd}{nn' - mm'}$, are the expressions for these magnitudes.

What values must we assign to the letters m, m', n, n', d, that the problems 73, 74, 75, 76, 77, may be comprised by the two last formulæ?

79) Two magnitudes whose difference $= d$, are for three reasons, of which it is assumed that they have no mutual influence on one another, as $m : n$, $m' : n'$, $m'' : n''$. How are these two magnitudes expressed?

$$\textit{Ans.}\ \frac{mm'm''d}{nn'n'' - mm'm''},\ \text{and}\ \frac{nn'n''d}{nn'n'' - mm'm''}.$$

80) But when the sum s of these magnitudes, instead of their difference d is given, how are they then expressed?

$$Ans.\ \frac{mm'm''s}{nn'n''+mm'm''}, \text{ and } \frac{nn'n''s}{nn'n''+mm'm''}.$$

81) A person puts a capital of $5500 out at interest, at 4 per cent, and 4½ years after, another capital of $8000, at 5 per cent. If he leaves these two capitals constantly at interest, in how many years will he have drawn the same interest from both?

Ans. In 10 years from the time he put out the first sum.

82) A person possesses a wagon with a mechanical contrivance, by which the difference of the number of revolutions of the wheels on a journey may be determined. It is known that each of the fore-wheels is 5¼, and that each of the hind-wheels is 7⅛ feet in circumference. Now, when in a journey the fore-wheel has made 2000 revolutions more than the hind-wheel, how great was the distance travelled?

Ans. 39900 feet.

83) When the fore-wheel of the above mentioned vehicle is a feet, and the hind-wheel b feet in circumference, how great will be the distance gone over, when the fore-wheel has made n revolutions more than the hind-wheel?

$$Ans.\ \frac{abn}{b-a} \text{ feet.}$$

84) A wine merchant has two kinds of wine; the one costs 9 shillings* per gallon, the other 5. He wishes to mix

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* These are New England shillings, of which 6 = $1.

both wines together, in such quantities, that he may have 50 gallons, and each gallon, without profit or loss, may be sold for 8 shillings. How much must he take of each sort to make up this mixture?

Ans. $37\frac{1}{2}$ gallons of the best, and $12\frac{1}{2}$ gallons of the other sort.

85) Let the price of the best wine in the preceding problem be a shillings, the price of the worst $= b$ shillings, the number of gallons that are to be produced by the mixture $= n$, and the price of the mixture $= c$. How much must be taken of each sort?

Ans. $\frac{(a-c)n}{a-b}$ gallons of the worst, and $\frac{(c-b)n}{a-b}$ of the best sort.

86) A goldsmith has two kinds of silver, viz. one kind contains 14 parts pure silver, (that is, each pound of which contains 14 ozs. of pure silver, and 2 ozs. of alloy), and the other 8.* He cannot, however, use that which is as good as the first, nor that which is so bad as the 2d; because he wishes to make a dish, which shall be 20 lbs. in weight, each pound of it containing 12 ozs. of pure silver. How many pounds must he take of each kind, so that by melting them together, he may obtain both the desired weight and the desired alloy?

Ans. $13\frac{1}{3}$ lbs. of the best, and $6\frac{2}{3}$ lbs. of the worst silver.

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* The pound of silver is on the continent of Europe divided into 16 ounces.—The convention money of Germany contains 13 parts pure silver, and 3 parts alloy.

87) A wine merchant has 40 bottles of wine, one of which he sells for 7 shillings; since, however, he deems this price too high for his customers, he wishes, as he intends to deal fairly, to add as much water to it, as will enable him to sell a bottle of the mixed wine for $1. How many bottles of water must he add to it?

Ans. $6\frac{2}{3}$ bottles.

88) A person has 35 lbs. of silver, each of which contains 15 out of 16 parts, pure, and he wishes to add so much copper to it, that the lb. may contain only 12 parts pure silver, to 4 parts alloy? How much copper must he add?

Ans. $8\frac{3}{4}$ lbs.

89) How much silver of 8 parts out of 16, pure, must be melt with $7\frac{1}{2}$ lbs. of 13 parts pure, that the contents may be brought to 9 parts pure?

Ans. 30 lbs.

90) A person wishes for 17 silver coins, consisting of four-pences and quarter-dollars, for $2,75. How many coins of each sort can he receive for that amount?

Ans. 8 four-pences and 9 quarter-dollar pieces.

What has this problem in common with the 84th and 86th? And how may their common property be expressed by words?

91) Required two numbers whose sum $= a$, and which are of such a nature, that if the 1st be multiplied by m, and the other by n, the sum of the products may be $= b$ What expressions will give these numbers?

Ans. $\frac{b - na}{m - n}, \frac{ma - b}{m - n}$.

What do these expressions mean when $m = n$? What, when at the same time, $b = na = ma$?

92) One of my acquaintances is now 40 years old, his son is 9 years. In how many years will this man, who is now more than 4 times as old as his son, be only twice as old as he?

Ans. In 22 years.

93) One of my acquaintances is now 30, his elder brother 20; and consequently 3:2 is the ratio of his age to that of his brother's. In how many years will their ages be as 5:4?

Ans. In 20 years.

94) But, on the other hand, how many years is it since he was 6 times as old as his brother?

Ans. 18 years ago.

95) He has, besides the above mentioned brother, another, who is now only 6 years old. When will the ages of both his brothers amount together to his own?

Ans. In 4 years.

96) His father is now 49 years old, and consequently the three brothers are together 7 years older than their father; there was a time, however, when the father's age was exactly equal to the sum of the ages of his three sons. How long is that ago?

Ans. 3½ years.

97) At one time his father said to him (the youngest son not being then born) that he was the 4th part older than both his sons together. How long is that ago?

Ans. 9 years.

98) Saltpetre and sulphur are mixed together in a mass of 80 lbs. and in such a proportion, that for every 7 parts of

saltpetre there are 3 parts of sulphur. How much saltpetre must still be added to the mass, so that the proportion of these substances may be such, that for every 11 parts of saltpetre, there may be 4 parts of sulphur?

Ans. 10 lbs.

99) On the other hand, how much sulphur must be taken away from the mass in order that the proportion of 11 : 4 may be obtained?

Ans. $3\frac{7}{11}$ lbs.

100) But if there be as much saltpetre added, as there is taken away of the sulphur, so that the weight of the whole mass remains unchanged; how much saltpetre must be added to the mass, and in its stead taken away from the sulphur, to have still the proportion of 11 : 4?

Ans. $2\frac{2}{3}$ lbs.

101) In a company there were, at first, 3 times as many gentlemen as ladies; but afterwards, when 8 gentlemen with their wives went away, the proportion of the persons present of both sexes was still more unequal, for there remained 5 times as many gentlemen as ladies. Of how many of each sex did the company at first consist?

Ans. Of 48 gentlemen and 16 ladies.

102) What numbers must be added to each of the two given numbers, a and b, that the sums may be as $m : n$? or, which means the same, to what number must a and b be added, that the sums may have the same proportion as $m : n$?

Ans. $\frac{mb - na}{n - m}$ is the number sought.

103) What number must be added to a, and subtracted from b, that the sum may be to the difference as $m : n$?

Ans. $\frac{mb - na}{m + n}$.

104) What number must be subtracted from a and b respectively, that the differences may be as $m : n$?

Ans. $\frac{na - mb}{n - m}$.

In these three last problems, the problems 92, 93, 94, 98, 99, 100, are contained as particular cases. What values must we assign to the quantities a, b, m, n, in order to adapt this formula to those cases?

105) In a full wine cask there are three bung-holes; by the first the wine can be drawn off in 2, by the 2d in 3, and by the third in 4 hours. What time will it take when all three bung-holes are spouting at once?

Ans. $55\frac{5}{13}$ minutes.

106) A cistern can be filled by three pipes; by the first in $1\frac{1}{3}$ hours, by the 2d in $3\frac{1}{3}$ hours, and by the third in 5 hours. In what time will this cistern be filled when all three pipes are open at once?

Ans. In 48 minutes.

107) In order to make the foregoing problem more general, let the time which the first pipe takes in filling alone the cistern $= a$, the time which the second takes in doing the same $= b$, and the time required by the third $= c$. What expression gives the time in which all three pipes together will fill it?

Ans. $\frac{abc}{ab + ac + bc}$.

108) What expression will give the time, in which four pipes will fill the cistern, when they singly fill it in the times, a, b, c, d, respectively?

Ans. $\dfrac{abcd}{abc + abd + acd + bcd}$.

109) Three masons are employed in building a wall. The first does 8 cubic feet in 5 days, the 2d 9 cubic feet in 4 days, and the 3d 10 cubic feet in 6 days. How much time will these masons need, when they work together, to build 756 cubic feet of the wall?

Ans. $137\frac{13}{331}$ days.

110) A mechanic can finish a certain piece of work expressed by a, in a space of time expressed by b; another mechanic can finish the work c, in the time d; and a third, the work e, in the time f. In what time will these three mechanics together accomplish the work g?

Ans. In the time $\dfrac{bdfg}{adf + bcf + bde}$.

111) A cistern of $755\frac{1}{4}$ cubic feet is to be filled by three pipes. The first discharges 12 cubic feet in $3\frac{1}{4}$ days, the 2d $15\frac{1}{3}$ cubic feet in $2\frac{1}{2}$ days, and the third 17 cubic feet in 3 days. In what time will the cistern be filled?

Ans. In $43\frac{3}{4}$ days.

112) Three causes respectively produce in the times t, t', t'', the effects e, e', e''. What time will be required, when all three causes operate together, in order to produce the effect E, supposing that they act independently of one another?

Ans. The time is $\dfrac{Ett't''}{et't'' + e'tt'' + e''tt'}$.

In this last, the problems 105, 106, 107, 108, 109, 110, 111, are contained. How?

113) A person has 3 pieces of metal of equal size. Of the first, 5 cubic inches weigh $69\frac{3}{4}$ ounces; of the 2d, $3\frac{1}{3}$ cubic inches weigh 41 ounces, and $4\frac{2}{7}$ cubic inches of the 3d weigh 91 ounces. If, then, the three pieces together weigh $949\frac{2}{3}$ ounces, what is the size of each?

Ans. 20 cubic inches.

114) A charitable man wished to collect some money in a numerous company for a poor person. Each of the company offering to contribute 75 cents, the collector considered it too much; because he would then collect $25 more than was required for the man. He therefore proposed that each of them should only give 30 cents; but, on calculation, found that it would be too little; because he then would fall short of the required sum by $20. How many persons did the company consist of? How much did the man need? And how much must each contribute in order to collect the sum required?

Ans. The company consisted of 100 persons, the sum which was to be collected was $50, and the contribution of each 50 cents.

115) A merchant, in order to pay a debt, is obliged to lower the price of a certain commodity to the prime cost. On account of bad book-keeping he neither knows the weight nor the prime cost of the article. He only remembers this much, that if he had sold the pound for $1,25, he would have gained $7,50 by it, and that if he had sold the pound for 90 cents, he would have lost $13,50 by it. What, according to these conditions, were the weight and the prime cost of the article?

Ans. The weight was 60 lbs., and the prime cost per lb. $\$1{,}12\frac{1}{2}$.

116) A person wishes to raffle away a gold watch, and for that purpose makes a certain number of tickets. If he sells the tickets for \$1,25, he would lose \$20, because the watch costs more than he would in this case get; but if he sells the tickets for \$1,62½, then he gains \$10. How much, therefore, does the watch cost him, and how many tickets did he sell?

Ans. The watch costs \$120, and he sold 80 tickets.

117) A master-mason has engaged a number of masons for the erection of a building. He finds, after entering into a calculation, that if he gave each man m shillings a day, he would daily expend one shilling less than was assigned for that purpose by the estimate; and that he would lose b shillings, if he should give each n shillings a day. How many men did he hire? And what was the fixed daily wages of each?

Ans. The number of masons was $\frac{a+b}{n-m}$, apn the daily wages $\frac{an+bm}{n-m}$ shillings.

118) If a certain number be multiplied by m and m', we obtain two products, which exceed a certain other number by a and a'; what are the two numbers?

Ans. The first is $\frac{a-a'}{m-m'}$, and the 2d $\frac{m'a-ma'}{m-m'}$.

What values must be assigned to the letters m, m', a, a', that the problems 114, 115, 116, 117, may be comprised in this formula?

119) Find a number which has this property, that if it be

multiplied by 5, it gives a product which is as much above 20, as the number itself is under 20. What number is it?

Ans. 6⅔.

120) In order to meet all my expenses, said a person, I ought to have a yearly income of $540; but it wants a considerable sum of this. Were my income 3½ times as great as it really is, then I should be able not only to meet my expenses, but even have as much above my income as I am now deficient. What was this man's yearly income?

Ans. $240.

121) A copyist was asked how many sheets he wrote weekly, and answered, "I only work 4 hours a day, and cannot finish 70 sheets, which I could wish to do; but if I could work 10 hours a day, then I should write weekly exactly as many above 70 sheets, as I now write less than this number.' How many sheets did he write weekly?

Ans. 40 sheets.

122) A person asked a surveyor how far the two signals, whose distance he had just measured, were from one another. He answered, "Their distance is far short of 1000 feet; for if I add to the distance its third part, increase this sum by 176, and multiply this last by 2½, then I shall as much exceed 1000, as the distance falls short of it." How far were the signals from one another?

Ans. 360 feet.

123) A person wished to buy a house, and in order to raise the requisite capital, he draws the same sum from each of his debtors. He tried, whether, if he obtained $250 from each, it would be sufficient for the purpose; he found, however, that he would then still lack $2000. He tried it,

therefore, with $340; but this gave him 880 more than he required. How many debtors had he? What was the sum to be raised? And how much must he exact from each debtor?

Ans. The number of his debtors is 32, the capital $10000, and the sum which he must exact from each $312½.

124) A merchant has to make the following payments at three different periods; $2832 in 3 months, $2560 in 9 months, and $1450 in 16 months. The creditor wishes to receive the whole sum of $6842 at once. When ought the payment to be made?

Ans. In 8 months.

125) A person wishes to make the following payments at 4 different periods; one sum a in l months, a sum b in m, a sum c in n, and a sum d in p months. If he wishes to pay his whole debt, $a + b + c + d$, at once, at what period must he do it?

Ans. In $\frac{al + bm + cn + dp}{a + b + c + d}$ months.

126) A capitalist engages to lend a merchant $16000, for 15 months; but as he is not able to furnish this sum at once, they agree to this, that the capitalist shall first furnish $5000, after the expiration of 6 months $3000 more, and then in 8 months the last $8000. How much longer can the merchant retain the same capital of $16000, so that neither party may suffer?

Ans. 9½ months.

128) A land proprietor entered into a contract with his neighbor, in which he bound himself to take 400 of his

neighbor's oxen into pasture. The neighbor, however, sent, with the land-owner's consent, at first only 200 head; in 7 months 250 more, and in 8 months after that, again 50 more. How much longer must the grazier feed all the 600 oxen, so as to fulfil the engagement he has entered into?

Ans. 2⅓ months.

128) A person purchases a certain commodity for $4500, which he is to pay for at the expiration of a year. He agrees with the seller to give him $1500 cash, and the remaining $3000 at 4 equal periods, $750 each time. What period must be fixed upon, so that neither party may be a loser?

Ans. A period of 7⅕ months.

129) A certain sum is to be paid as follows: $1376 in 5 months; 3 months later $2560, and the remainder 5 months after this. If the whole sum be paid at once, it must take place in 10 months. What was the sum to be paid?

Ans. $7936.

130) A person is to pay $7000; viz. $2000 in 3½ months, $3500 in 4 months, and $1500 in 14 months. His creditors propose to him to pay this sum in two different instalments, the half each time, so that the 2d payment may be a month later than the first. If the debtor agrees to this, in what time must the first instalment be paid?

Ans. In 5½ months.

131) Two persons bid for a garden which is to be sold. The one offers $7705; viz. $3365 cash, and $4340 in 8 years; or if the seller wish it, he will give this last sum also in cash, deducting 5 per cent discount. The 2d bids another sum, which, calculated from this time, is to be paid in 3 equal

instalments, each at an interval of 2 years; viz. the 1st third part in 2 years, the 2d in 4 years, and the last third in 6 years; or, if the seller wish it, cash immediately, deducting 5 per cent discount. The seller considers both offers the same. How much did the 2d bid?

Ans. \$7722.

132) Three merchants, A, B, C, enter into partnership. A advances \$1200, B \$800, and C \$600. A leaves his money 8 months, B 10 months, and C 14 months in the business. They gain \$500. What is the share of each?

Ans. A receives \$$184\frac{8}{13}$, B \$$153\frac{11}{13}$, C \$$161\frac{7}{13}$.

133) Let A's capital (in the preceding problem) be equal to a, B's $= b$, C's $= c$; A leaves his money l months, B m months, and C, n months in the business. Let the profits $= g$. What share does each receive of the profits?

$$\textit{Ans.}\ \text{A's share is} = \frac{alg}{al + bm + cn},$$

$$\text{B's} = \frac{bmg}{al + bm + cn},\ \text{and C's} = \frac{cng}{al + bm + cn}.$$

134) Three merchants entered into trade. The first contributed \$17000, the 2d \$13000, and the 3d \$10000. As they must have some person to conduct the business, the one who had furnished the least offered to undertake the management of it, but with this condition, that he receives 3 per cent profit, besides what he is entitled to from his deposited capital. Now, it is found that they have gained \$$35262\frac{1}{2}$. How much is due to each?

Ans. To the first \$14875, to the 2d \$11375, to the 3d \$$9012\frac{1}{2}$.

135) Three creditors claim an estate, which, after deducting the proper expenses, amounts to \$3139; the claim of the first was \$2000, the 2d \$2500, and that of the 3d \$3500. But as the estate is not sufficient to satisfy all the creditors, and, moreover, their claims are not equally legal, the sum is, by a judicial decree, to be divided amongst the creditors in proportion to their demands; yet, for the alleged reason, the 2d receives 10, and the 3d, 25 per cent more than his share. How much, therefore, does each receive?

Ans. The 1st \$688, the 2d \$946, the 3d \$1505.

136) Let the estate in the preceding problem be $= a$, the claims of the 3 creditors, f, g, h; let the 2d receive m per cent, and the 3d n per cent more than the first. How much does each receive?

Ans. The 1st $\dfrac{100af}{100(f+g+h)+gm+hn}$,

The 2d $\dfrac{(100+m)ag}{100(f+g+h)+gm+hn}$,

The 3d $\dfrac{(100+n)ah}{100(f+g+h)+gm+hn}$.

How must these formulæ be altered, if the 2d, instead of receiving m per cent more, should get m per cent less?

137) Three persons, A, B, C, contribute a certain sum for business. B contributes $\frac{1}{2}$ more than A, and C \$300 more than A and B together. After some time the profit, amounting to \$5020, is divided, and C receives \$2570 for his share. How much did each contribute?

Ans. A \$2450, B \$3675, C \$6425.

138) Three merchants, A, B, and C, form themselves into a trading company; C contributed \$5600; but A \$320 less

than B; A leaves his money 7 months, B 14 months, and C 12 months in the business. The profits, amounting to \$2402$\frac{1}{6}$, are divided amongst the partners, in proportion to the sum advanced, and the time. Now, B receives \$879$\frac{2}{3}$ for his share. How much did A and B advance?

Ans. A \$3450, B \$3770.

139) A person dies, and leaves 4 sons and a fortune of \$1100. Ten months after his death the will was first opened, and in this time the children had expended all the property they had inherited, together with the interest of it. At the same rate of expenditure, and the same rate of interest, 3 other children spent a capital of \$1200 in 15 months. How high were the interests of these sums calculated? And how long would it take 6 children, under the same circumstances, to expend \$1650?

Ans. The interest amounted monthly to $\frac{2}{3}$ per cent, and it would take 6 children 10 months to spend the \$1650.

140) Five brothers, in the space of 9 months, squandered a capital of \$4800, together with the interest of it for this whole time. At the same rate of expenditure, 2 other persons squandered \$3320, with interest, in 16 months. The rate of interest was in both cases the same. How much did each spend monthly?

Ans. \$110$\frac{2}{3}$.

141) A servant received from his master \$40 wages, yearly, and a suit of livery. After he had served 5 months he asked for his discharge, and received for this time, the livery, together with \$6$\frac{1}{2}$ in money. How much did the livery cost?

Ans. \$18.

142) A farmer employs 2 journeymen, who work for him at the same rate. To one he gave 16 bushels of rye and \$14 for 56 days' work; and the other 30 bushels of rye and \$17$\frac{1}{4}$ for 84 days' labor. What did he call a bushel of rye?

Ans. 62$\frac{1}{2}$ cents.

143) A master hired a journeyman, and promised him 8 shillings for each day that he worked for him; but if he worked anywhere else, then the journeyman must pay him 5 shillings daily for his board. At the expiration of 50 days they settle, aud the journeyman receives 36 dollars and 2 shillings. How many days has he worked for his master?

Ans. 36 days.

144) A farmer brought a basket of eggs to market, and offered them for sale at 7 cents a dozen. A person who was passing by, stumbled against the basket and broke 5 dozen of his eggs. Being paid for them, he resolved to sell the remainder at 8 cents a dozen; because he would then get as much by them, as he would have received before by his full number, besides the money for the eggs that were broken, which would then be clear profit. How many dozen eggs did the farmer bring to market?

Ans. 40.

145) A cook, who was carrying lemons, was asked how many he had. As he was a good arithmetician, he answered in the following way: These lemons cost me 75 cents a dozen; but if I had got the 5 lemons into the bargain, which I asked for, they would have cost me 10$\frac{5}{12}$ cents less per dozen. How many lemons did he carry?

Ans. 31.

146) A merchant sends for a piece of cloth, for which he

pays \$$2\frac{1}{2}$ a yard. In measuring it over a second time, he finds that it contains in fact 5 yards more than were charged him; but, at the same time, of so bad a quality, that for want of opportunity to send it back, he is obliged to sell it at \$2 a yard. He calculates that in doing this he loses $13\frac{1}{3}$ per cent in the whole bargain. How many yards did the piece contain?

Ans. By statement 60, but in fact 65.

147) Now, says a person, I spend a 7th part of my income at the theatre, and the rest in my ordinary expenses; could I, however, receive an addition of \$100 yearly, then I might spend a 5th of my annual income at the theatre, and still have \$40 more than before, for my ordinary expenses. What was his income?

Ans. \$700.

148) An admirer of the sciences, who till now, had expended the fourth part of his yearly income in the purchase of books, resolves, on account of an addition which he has received to it, to lay out, in future, the third part of it in this way; because he calculates, that notwithstanding this increase, he will still have as much left for his ordinary expenses as before. What proportion did this addition bear to his former income?

Ans. It amounted to the 8th part of it.

149) In a certain town each house-keeper contributed formerly the 7th part of his house-rent for taxes; afterwards the tax was raised, and each was obliged to pay the 6th part. How much must each house-keeper raise his lodgers, in order that he may receive as much after paying the tax, as he did before?

Ans. The 35th part of the former rent.

150) I once had an untold sum of money lying before me. From this I first took away the 3d part, and put in its stead $50. A short time after I took from the sum thus augmented, the 4th part, and put again in its stead $70. I then counted my money and found $120. What was the original sum?

Ans. $25.

151) From a sum of money $50 more than the half of it is first taken away; from the remainder $30 more than its fifth part; and again, from the 2d remainder $20 more than its fourth part. At last there remained only $10. What was the original sum?

Ans. $275.

152) A rich man bequeathed in his will a certain sum, which was to be divided amongst three of his domestics as follows: The valet's share is $200, together with half of the remainder; the cook's the 5th part of the residue and $400. The remainder, which amounts to $520, belongs to the coachman. How much did the legacy amount to?

Ans. $2500.

153) A countryman brings his eggs to market, and first sells 4 more than the half of them, then he goes further, and sells half of the remainder and 2 over. Now, 6 eggs more than half of the remainder are stolen from him, and dissatisfied about this loss, he returns to his village with the 2 eggs which remained in his basket. How many eggs did he take to town?

Ans. 80.

154) A merchant adds yearly to his capital one third, but takes from it, at the end of each year, $1000, for his expen-

ses. At the end of the third year, after deducting the last $1000, he finds himself in possession of twice the sum he had at first. How much did he possess originally?

Ans. $11100.

155) A merchant increases his capital yearly by 20 per cent, but takes from it every year $1000 for the support of himself and family. After he had carried on his business, in this manner, for 3 years, he finds, after deducting the usual sum of $1000, that his capital has increased $200 more than $\frac{3}{5}$ of the original sum. What was his original capital?

Ans. $30000.

156) A man, to please his children, brings home a number of apples, and divides them as follows: To the first and eldest of his children he gives the half of the whole number, less 8; to the second the half of the remainder, again diminished by 8; and he does the same with the third and fourth. After this he gives the 20 remaining apples to the fifth. How many apples did he bring home?

Ans. 80.

157) I take a certain number, multiply it by $3\frac{3}{7}$, take 60 from the product, multiply the remainder again by $2\frac{1}{2}$, and subtract again 30, when nothing remains. What is the number?

Ans. 21.

158) A spendthrift lends his fortune at 4 per cent interest. After he had let it remain for 2 years, he took out the fourth part of it, and allowed the remainder to stand 7 months. At the expiration of this time he took, once more, the fourth part of the remainder, and allowed the capital, thus diminished, to

stand 13 months, when he demanded all his remaining fortune. In this space of 44 months he had drawn no less than \$6093¾ of interest money. What was his fortune at first?

Ans. \$50000.

159) A father leaves a number of children, and a certain sum, which they are to divide amongst them as follows: The first is to receive \$100, and then the 10th part of the remainder; after this, the second has \$200, and the 10th part of the residue; again, the third receives \$300, and the 10th part of the remainder; and so on, each succeeding child is to receive \$100 more than the one immediately preceding, and then the 10th part of that which still remains. At last it is found that all the children have received the same. What was the fortune left? And how many children were there alive?

Ans. The fortune left was \$8100, and the number of children 9.

160) But what must the sum left, and the number of children be, when the first child receives \$30, and the 9th part of the remainder; the 2d \$60, together with the 9th part of the remainder, and so on, each succeeding one \$30 more than the one immediately preceding, together with the 9th part of the residue, and yet all have the same sum?

Ans. \$1920, and 8 children.

161) Further, what must the fortune and number of children be, when, in general, the first receives a dollars, together with the nth part of the remainder; and each succeeding child a dollars more, together with the nth part of the remainder, and it is found, at last, that they have all received the same?

Ans. The fortune $= (n - 1)^2 a$, the number of children $= n - 1$.

162) A general, wishing to draw up his regiment into a square, tried it in two ways. The first time he had 39 men over; the 2d, having extended the side of the square by 1 man, he wanted 50 men to complete the square. What was the number of soldiers in the regiment?

Ans. 1975 men.

163) A person has a certain number of dollars, which he wishes to arrange in the form of a square. At the first trial he had 130 dollars over; but when he enlarged the side of the square by 3 dollars, there only remained 31. How many dollars had he?

Ans. 355.

164) It is required to find a number, such, that if the two numbers, a and b, are added to it, the difference between the squares of these sums may be $= d$. What number is it?

Ans. $\dfrac{d - a^2 + b^2}{2(a - b)}$.

Are the two preceding problems included in this one?

165) Determine the size of 3 wine casks from the following conditions: When the first empty cask is filled from the 2d full cask, there only remains in the 2d $\frac{2}{9}$ of the wine; if the 2d empty cask is filled from the 3d full one, then there remains in the 3d only $\frac{1}{4}$ of the wine; but if he wishes to fill the 3d empty cask from the first full one, there would not be enough by 50 gallons. How many gallons does each cask contain?

Ans. The first 70, the 2d 90, the 3d 120 gallons.

166) A person has 4 wine casks of different sizes. When he fills the 2d empty cask from the first full one, there re-

mains in the first only $\frac{4}{7}$ of the wine; when he fills the 3d empty cask from the 2d full one, then there is left in the 2d only $\frac{1}{4}$ of the wine; but when he attempts to fill the 4th empty cask from the 3d full one, then only $\frac{9}{16}$ of the 4th is filled, and if he wished to fill the 3d and 4th empty casks from the first full one, then these would not only be filled, but he would have 15 gallons over. How many gallons, then, does each of these 4 casks contain?

Ans. The first 140, the 2d 60, the 3d 45, and the 4th 80 gallons.

XVI. PROBLEMS FOR SIMPLE EQUATIONS, WITH MORE UNKNOWN QUANTITIES.

1) Required two numbers, whose sum is 70, and whose difference is 16?

Ans. 43 and 27.

2) Required to find two numbers whose sum $= a$, and whose difference $= b$. How may these numbers be expressed?

Ans. The first $\frac{a + b}{2}$, the other $= \frac{a - b}{2}$.

3) Two purses, together, contain $300. If you take $30 out of the first and put them into the 2d, then there is the same in each. How many dollars does each contain?

Ans. The first $180, the 2d $120.

4) A says to B, give me $100, and I shall have as much

as you. No, says B to A, give *me* rather $100, and then I shall have twice as much as you. How many dollars has each?

Ans. A $500, and B $700.

5) A person has two snuff-boxes. If he put $8 into the first, then it is half as valuable as the other; but if he takes $8 out of the first, and puts them into the 2d, then the latter is worth three times as much as the former. What is the value of each?

Ans. The first is worth $24, the 2d $64.

6) A and B possess together a fortune of $570. If A's fortune were 3 times, and B's 5 times as great as each really is, then they would have together $2350. How much was each?

Ans. A $250, B $320.

7) Find two numbers of the following properties: When the one is multiplied by 2, the other by 5, and both products added together, the sum is $= 31$; on the other hand, if the first be multiplied by 7, and the 2d by 4, and both products added together, we shall obtain 68.

Ans. The first is 8, and the 2d is 3.

8) If one of two numbers is multiplied by a, the other by b, the sum of the products is $= k$; but if the first be multiplied by a', and the 2d by b', then the sum of the products $= k'$. How can these numbers be expressed?

Ans. $\dfrac{b'k - bk'}{ab' - a'b}$, $\dfrac{ak' - a'k}{ab' - a'b}$.

9) Two numbers are given by the following data: If the first be increased by 4, it will be $3\frac{1}{4}$ as great as the 2d; but

if the 2d be increased by 8, then it will be half as great as the first. What are the two numbers?

Ans. 48 and 16.

10) When the first of two numbers is increased by a, it becomes m times as great as the 2d; but when the 2d is increased by b, then it becomes n times as great as the first. How are these numbers expressed?

$$\textit{Ans.}\ \text{The first} = \frac{a + mb}{mn - 1},\ \text{the 2d} = \frac{b + na}{mn - 1}.$$

11) Said a man to his father, "How old are we?" "Six years ago," answered the latter, "I was one third more than three times as old as you; but three years *hence* I shall be obliged to multiply your age by $2\frac{1}{6}$ in order to obtain my own." What is the age of each?

Ans. The father 36, the son 15 years.

12) A and B, jointly, have a fortune of $9800. A invests the 6th part of his property in business, and B the 5th part, and still each has the same sum remaining. How much has each?

Ans. A has $4800, B $5000.

13) A owes $1200, B $2500; but neither has enough to pay his debts. Lend me, said A to B, the 8th part of your fortune, and I shall be enabled to pay my debts. B answered, I can discharge my debts, if you will lend me the 6th part of yours. What was the fortune of each?

Ans. A's fortune is $900, and that of B $2400.

14) A capitalist borrows $8000 on favorable conditions, because he has an opportunity of laying out $23000, at higher interest, and he has an overplus of $905 of interest,

yearly. On the same conditions he borrows, on the one hand, $9400, and on the other lends $17500; this brings him an overplus of $539½ in interest, yearly. At what rate of interest did he borrow and lend money?

Ans. At 4½ and 5½ per cent.

15) A person has two large pieces of iron whose weight is required. It is known that ⅖ of the first piece weigh 96 lbs. less than ¾ of the other piece; and that ⅝ of the other piece weigh exactly as much as 4⁄9 of the first. How much did each of these pieces weigh?

Ans. The first weighs 720, the 2d 512 lbs.

16) A cistern containing 210 buckets, may be filled by 2 pipes. By an experiment, in which the first was open 4, and the 2d 5 hours, 90 buckets of water were obtained. By another experiment, when the first was open 7, and the other 3½ hours, 126 buckets were obtained. How many buckets does each pipe discharge in an hour? And in what time will the cistern be full, when the water flows from both pipes at once?

Ans. The first pipe discharges 15, and the 2d, 6 buckets; it will require 10 hours for them to fill the cistern.

17) A person has 500 coins, consisting of sovereigns and shillings; their value amounts to 334£, 14*s*. How many has he of each coin.

Ans. 326 sovereigns, 174 shillings.

18) A person has two kinds of goods. 8 lbs. of the first, and 9 of the 2d, cost together $18,46; farther, 20 lbs. of the first, and 16 of the 2d, cost together $36,40. How much does the lb. of each article cost?

Ans. 62 cents, and $1,50.

19) 15 Silesian and 33 Leipzig ells together are equal to $39\frac{1}{2}$ Brabant ells; farther, 24 Silesian and 55 Leipzig ells are together equal to 65 Brabant ells. What proportion, according to these conditions, do the Silesian and Leipzig ells bear to the Brabant? Farther, what proportion does the Silesian ell bear to the Leipzig ell? And by how many per cent do the two last differ from one another?

Ans. The Silesian ell is to the Brabant ell as 5 to 6, the Leipzig to the Brabant as 9 to 11, the Silesian to the Leipzig as 55 to 54; and the Silesian ell is $1\frac{25}{27}$ per cent longer than the Leipzig.

20) $17\frac{1}{2}$ Dantzig and 19 Berlin feet, make together as much as $34\frac{3}{4}$ Rhenish feet; farther, 5 Dantzig and $9\frac{1}{2}$ Berlin feet are equal to $13\frac{5}{56}$ Rhenish. According to these statements, what proportion do the Dantzig and the Berlin feet bear to the Rhenish. What proportion does the Dantzig foot bear to the Berlin? And by how many per cent do these last two differ?

Ans. The Dantzig foot is to the Rhenish as 32 to 35, the Berlin to the Rhenish as 75 to 76, the Dantzig to the Berlin as 2432 to 2625, and the Berlin foot is $7\frac{569}{608}$, or about $7\frac{14}{15}$ per cent larger than the Dantzig.

21) 40 French miles, when reduced to Geographical or German miles, amount to $12\frac{1}{2}$ such miles more than 53 English; 10 French, and $26\frac{1}{2}$ English miles are together equivalent to $11\frac{3}{4}$ German miles. At this rate, what proportion do the French and English bear to the German? And what relation have the French to English?

Ans. The French mile is to the German as 3 to 5,

the English to the German as 23 to 106, and
the French to the English as 318 to 115.

22) A traveller said, "I have travelled through Germany, France and England, and have expended in these three countries 8325 dollars, viz. in Germany 1520 rix dollars, in France 7540 francs, and in England 820 pound sterling." As he was asked to fix by this calculation the value of the different kinds of money, he gave for answer, that 5 pound sterling amounted to 3 rix dollars more than 108 francs. What was each coin computed at?

Ans. The pound sterling at 6 rix dollars, and the franc at $\frac{1}{4}$ rix dollar.

23) A person has two horses and two saddles, one of which cost \$50, the other \$2. If he places the best upon the first horse, and the worst upon the 2d, then the latter is worth \$8 less than the other; but if he puts the worst saddle upon the first, and the best upon the 2d horse, then the latter is worth $3\frac{3}{4}$ times as much as the former. What is the value of each horse?

Ans. The first \$30, the 2d \$70.

24) There is a fraction such, that if 1 be added to the numerator, its value $= \frac{1}{3}$, and if 1 be added to the denominator, its value $= \frac{1}{4}$. What fraction is it?

Ans. $\frac{4}{15}$.

25) It is required to find a fraction such, that if 3 be subtracted from the numerator and denominator, it is changed into $\frac{1}{4}$, and if 5 be added to the numerator and denominator, it becomes $\frac{1}{2}$. What fraction is it?

Ans $\frac{7}{19}$.

26) B has lent out at interest $12600 more than A, and obtains 1 per cent more for his money, on account of which his interest amounts yearly to $730 more than A's. C has lent $3000 more than A, and at 2 per cent higher interest, wherefore he obtains for his sum, $380 yearly interest more than A. How much has each lent? And at what interest?

Ans. A has lent $10000, B 22600, C $13000.
A at 4 per cent, B at 5, and C at 6.

27) A company expended a certain sum at an inn, which is to be equally divided amongst them. Had there been 5 persons more, and had each spent $12\frac{1}{2}$ cents more, then the bill would have been $\$10,17\frac{1}{2}$; but had there been 3 persons less, and had each expended 5 cents less, the bill would have been $8,25. How many were there in the company? and what did each spend?

Ans. There were 11 persons in the company, and each spent 80 cents.

28) A work is to be printed, that each page may contain a certain number of lines, and each line a certain number of letters. If we wished each page to contain 3 lines more, and each line 4 letters more, then there would be 224 letters more on each page; but if we wished to have 2 lines less in a page, and 3 letters less in each line, then each page would contain 145 letters less. How many lines are there in each page? And how many letters in each line?

Ans. 29 lines per page, and 32 letters in a line.

29) Required to find two numbers, such, that if one be increased by a, and the other by b, the product of these two sums exceeds the product of the two numbers themselves by c; if, on the other hand, the one be increased by a', and the

other by b', the product of these sums exceeds the products of the two numbers themselves by c'. How are these numbers expressed?

Ans. They are $\dfrac{a'c - ac' + aa'(b' - b)}{a'b - ab'}$,

and $\dfrac{bc' - b'c + bb'(a - a')}{a'b - ab'}$.

Are the three preceding problems included in this one? And what values must be assigned to the letters, a, b, c, a', b', c', so that the former problems may be solved?

30) Not long ago, says a person, the barrel of wheat was by $1, and the barrel of rye 75 cents cheaper than they are now; then the price of the wheat was double that of the rye; their present prices are as 20 : 11. What is the price of each grain?

Ans. The wheat is $5, and the rye $2,75 per barrel.

31) A person had two barrels and a certain quantity of wine in each. In order to have an *equal* quantity in each, he poured out as much of the first cask into the second, as it already contained; then, again, he poured out as much of the 2d into the first as it then contained, and lastly he poured out again, as much from the first into the 2d, as there was still remaining in it. At last he had 16 gallons of wine in each cask. How many gallons did they contain originally?

Ans. The first 22, the 2d 10 gallons.

32) When in the preceding problem, there remained at last, a gallons in each barrel, how many gallons must they have originally contained?

Ans. The first $\frac{11}{8}a$, the 2d $\frac{5}{8}a$.

33) A wine merchant has two kinds of wine. If he mix

3 gallons of the worst with 5 of the best, the mixture is worth \$1 per gallon; but if he mixes 3¼ gallons of the worst with 8¾ gallons of the best, then the mixture is worth \$1,03⅓ per gallon. What does each wine cost per gallon?

Ans. The best wine \$1,12 per gallon, the worst 80 cents.

34) In general, let *a* gallons of the first wine, mixed with *b* gallons of the 2d, be worth *c* dollars per gallon; farther, let *f* gallons of the first, mixed with *g* gallons of the 2d, be worth *h* dollars per gallon. What does each wine cost per gallon?

Ans. The price of the first is $\frac{(a+b)\,cg-(f+g)\,bh}{ag-bf}$ dollars; the price of the 2d is $\frac{(a+b)\,cf-(f+g)\,ah}{bf-ag}$.

35) 37 lbs. of tin lose 5 lbs. when plunged into water, and 23 lbs. of lead lose 2 lbs. in water. A composition of tin and lead weighing 120 lbs. loses 14 lbs. in water.* How much does this composition contain of each metal?

Ans. 71 lbs. of tin, and 46 lbs. of lead.

36) 21 lbs. of silver lose 2 lbs. in water, and 9 lbs. of copper lose 1 lb. in water. Now, if a composition of silver and copper weighing 148 lbs. loses 14⅔ lbs. in water, how many lbs. does it contain of each metal?

Ans. 112 lbs. of silver, and 36 lbs. of copper.

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* How this is to be understood, and how upon it the theory of specific gravity is founded, may be explained by the teacher; as it is one of the most elementary principles of natural philosophy.

37) A given piece of metal, which weighs p pounds, loses a pounds in water. This piece, however, is composed of two other metals, which call A and B; of these we know that p lbs. of A lose b lbs. in water, and p lbs. of B lose c lbs. How much does this piece contain of each metal?

Ans. $\frac{(c-a)p}{c-b}$ lbs. of A, and $\frac{(a-b)p}{c-b}$ lbs. of B.

38) According to Vitruvius, Hiero, king of Syracuse's crown weighed 20 lbs., and lost $1\frac{1}{4}$ lbs., nearly, in water. Let it be assumed that it consisted of gold and silver only, and that 19,64 lbs. of gold lose 1 lb. in water, and 10,5 lbs. of silver, in like manner, lose 1 lb. How much gold, and how much silver did this crown contain?

Ans. 14,77.... lbs. of gold, and 5,22... lbs. of silver.

Is this problem contained in the preceding one? and what values must we here substitute instead of a, b, c, p?

39) Lead is 11,324 heavier than water; cork only weighs 0,24 times as much as water; again, fir weighs 0,45 as much as water. A person wishes to combine a piece of lead with a piece of cork, so that he may obtain a body of 80 lbs. weight, and which shall lose in water as much as 80 lbs. of fir would, so that it may float. How much lead and how much cork must be combined together?

Ans. 38,14 lbs. of lead with 41,85.... lbs. of cork.

40) Two different kinds of matter, one of which is p, and the other p' as heavy as water, are to be so united, that the body which arises from their union, may, on an average, be

p'' times as heavy as water, and weigh q lbs. How many lbs. must be taken of each kind?

Ans. $\frac{qp(p' - p'')}{p''(p' - p)}$ lbs. of the first,

and $\frac{qp'(p'' - p)}{p''(p' - p)}$ lbs. of the 2d.

Between what limits must p'' lie, that this problem, as it is here given, may be possible?

41) It is required to find two numbers, whose difference, sum and product, are to one another as $2:3:5$, consequently, whose difference is to their sum as 2 to 3, and whose sum is to their product as $3:5$. What numbers are they?

Ans. 2 and 10.

42) Find two numbers, whose sum is $= m$, and whose product is n times as great as their difference. What numbers are they?

Ans. $\frac{2n}{m-1}$, and $\frac{2n}{m+1}$.

43) Let the sum of two numbers be 13, and the difference of their squares 39. What numbers are they?

Ans. 5 and 8.

44) The sum of two numbers is $= a$, the difference of their squares $= b$. What are the numbers?

Ans. $\frac{a^2 + b}{2a}$, $\frac{a^2 - b}{2a}$.

45) The sum of two numbers is $= a$, the quotient arising from the division of the one by the other is $= b$. Find these numbers?

Ans. $\frac{a}{b+1}$, and $\frac{ab}{b+1}$.

46) A person was asked his own, his father's and his grand-father's ages. He answered, my age and my father's together, amount to 56 years, my father's age and my grand-father's to 100 years, mine and my grand-father's to 80 years. What was the age of each?

Ans. He himself is 18 years, his father 38, and his grand-father 62.

47) The sums of three numbers, taken two and two, are a, b, c. What are these numbers?

Ans. $\frac{a+b-c}{2}$, $\frac{a+c-b}{2}$, $\frac{b+c-a}{2}$.

48) A, B, C, owe together $2190, and none of them can alone pay this sum; but when they unite, it can be done in the following ways: 1st, by B's putting $\frac{3}{7}$ of his property to all A's; 2dly, by C's putting $\frac{5}{9}$ of his property to that of B's; or, by A's adding $\frac{2}{3}$ of his property to that of C. How much did each possess?

Ans. A $1530, B $1540, and C $1170.

49) A and B possess, together, only $\frac{2}{3}$ of the property of C; B and C have, together, 6 times as much as A; were B $680 richer than he actually is, then he would have as much as A and C together. How much has each?

Ans. A has $200, B $360, and C $840.

50) I have 3 purses lying before me, each of which contains a certain sum of money. If I take $20 out of the first, and put them into the 2d, then the latter contains 4 times as much as the former. If, on the other hand, I take $60 out of the 2d, and put them into the 3d, then this contains $1\frac{3}{4}$ times as much as there remains in the 2d. Again, if I take $40 out of the 3d, and put them into the first, then the 3d con-

tains $2\frac{7}{8}$ times as much as the first. What were the contents of each purse?

Ans. In the first there were $120, in the 2d $380, and in the 3d $500.

51) A, B, C, compare their fortunes. A says to B, give me $700 of your money, and I shall have twice as much as you retain; B says to C, give me $1400, and I shall have thrice as much as you have remaining; C says to A, give me $420, and then I shall have 5 times as much as you retain. How much has each?

Ans. A $980, B $1540, C $2380.

52) Find three numbers of the following properties: If 4 are subtracted from the first, and 4 added to the 2d, then the remainder is to the sum as 1 : 2; if 10 be subtracted from the 2d, and as many added to the 3d, then the remainder is to the sum as 3 : 10; but if 5 be subtracted from the first, and this number be added to the 3d, then the remainder is to the sum as 3 : 11. What numbers are they?

Ans. 20, 28, and 50.

53) A, B, C, together, possess $1820. If B give A $200 of his money, then A will have $160 more than B; but if B receive $70 from C, then both will have the same sum. How much has each?

Ans. A $400, B $640, and C $780.

54) Three persons, jointly, spent a certain sum; but none of them is alone, able to pay it. A, therefore, says to B, give me the 4th part of your money, and then I can pay it alone; B says to C, give me the 8th part of your money, and then I can also pay it; then says C to A, I shall also be able to pay it, if I receive from you half your money, although I, at pre-

sent, only possess \$4. How much did they spend? And how much had A and B?

Ans. They spent \$6½; A has \$5, and B \$6.

55) A person has three different pieces of silver of different alloy, viz. 15, 10, and 9 parts out of 16 pure. If he melt the 15 with the 10, then there will arise a composition of 11⅔ parts pure. The silver arising from the mixing the 15 and 9 together will be of the same purity. All three pieces together weigh 34 lbs. (at 16 ounces each). How much does each piece weigh by itself?

Ans. The 15 weighs eight lbs., the 10, sixteen, and the 9, ten lbs.

56) A person has three magazines, each of which contains three kinds of grain, viz. wheat, rye, and barley. The first ware-house contains 192 barrels of wheat, 63 barrels of rye, and 120 barrels of barley; the 2d 72 barrels of wheat, 210 barrels of rye, and 168 barrels of barley; the third 144 barrels of wheat, 162 barrels of rye, and 312 barrels of barley. The value of the first ware-house is \$2202; the value of the 2d \$2436, and the value of the third \$3228. What was the value of a barrel of each grain?

Ans. The barrel of wheat is worth \$7, the barrel of rye \$6, and the barrel of barley \$4.

57) A, B, C, purchase coffee, sugar and tea, at the same prices. A pays \$11,62½ for 7½ lbs. of coffee, 3 lbs. of sugar, and 2½ lbs. of tea; B pays \$16,25 for 9 lbs. of coffee, 7 lbs. of sugar, and 3 lbs. of tea; C pays \$12,25 for 2 lbs. of coffee, 5½ lbs. of sugar, and 4 lbs. of tea. What is the price of each per lb.?

Ans. The coffee 75 cents, the sugar 50 cents, and the tea \$2.

58) Three masons, A, B, C, are to build a wall. A and B, jointly, could build this wall in 12 days; B and C could accomplish it in 20 days; but A and C would do it in 15 days. What time would each take to do it alone in? And in what time will they finish it, if all three work together?

Ans. A requires 20 days, B 30, and C 60; all three together require 10 days.

59) Three laborers are employed in a certain work. A and B would, together, complete this work in a days; A and C require b days; but B and C, only c days. What time would each require, singly, to accomplish it in, supposing, under all circumstances, that each does the same quantity of work? And in what time would they finish it, if they all three worked together?

Ans. A requires $\frac{2abc}{bc + ac - ab}$ days,

B $\frac{2abc}{bc + ab - ac}$ days, and C $\frac{2abc}{ab + ac - bc}$ days,

Jointly, they require $\frac{2abc}{ab + ac + bc}$ days.

60) A cistern may be filled by three pipes, A, B, C. By the pipes A and B, it could be filled in 70 minutes; by the pipes A and C, in 84 minutes; and by the pipes B and C, in 140 minutes. What time will each pipe take to do it in? And in what time will the reservoir be filled, if the water flow into it from all the three pipes at once?

Ans. A requires 105 minutes, B 210, and C 420; all three pipes together require an hour.

Is this problem also included in the 59th; and, if so, how must the conditions be changed in the last, in order that it may be quite similar?

61) A person has 3 pieces of metal, each of which consists of gold, silver, and copper. The first contains 5 ozs. of gold, 15 ozs. of silver, and 30 ozs. of copper; the 2d contains 20 ozs. of gold, 28 ozs. of silver, and 48 ozs. of copper; the third contains 12 ozs. of gold, 39 ozs. of silver, and 24 ozs. of copper. Now, he wishes to take some from each, to melt into a mass, in order to obtain a composition, consisting of 10 ozs. of gold, 23 ozs. of silver, and 26 ozs. of copper. How much, then, must he take of each?

Ans. Of the first 10 ozs., of the 2d 24 ozs., and of the third 25 ozs.

62) Three soldiers, in a battle, make $96 booty, which they wish to share equally. In order to do this A, who made most, gives A and B as much as they already had; in the same manner, B then divided with A and C, and after this, C with A and B. If, then, by these means, the intended equal division is effected, how much booty did each soldier make?

Ans. A $52, B $28, and C $16.

63) In the three drawers of my cupboard I have a sum of $162. In order that there may be the same sum in all three drawers, I take out of the first as much as is necessary, to put into each of the other two, the half of what they already contained. I then take out of the 2d, and afterwards out of the 3d drawer, and put, each time, into the two other drawers the half of what they already contained. If by these means I have actually attained my object, how much did each drawer contain at first?

Ans. In the first there were $70, in the 2d 52, and in the 3d $40.

64) A, B, C, play faro. In the first game A has the bank, B and C stake the third part of their money, and win. In the 2d game B has the bank, A and C stake the third part of their money, and also win. Then C takes the bank, A and B stake the third part of their money, and again win. After the third game they count their money, and find that they all have the same sum; for each had 64 ducats. How much had each before they began to play?

Ans. A had 75 ducats, B 63, and C 54.

65) A, B, C, D, E, play together on this condition, that he who loses shall give to all the rest as much as they already have. First A loses, then B, then C, then D, and at last also E. All lose in turn, and yet at the end of the 5th game they all have the same sum, viz. each $32. How much had each before they began to play?

Ans. A had $81, B $41, C $21, D $11, E $6.

66) $2652 are to be divided amongst three regiments, in such a way, that each man of that regiment which fought best, shall receive $1, and the remainder is to be divided equally among the men of the two other regiments. Were the dollar adjudged to each man in the 1st regiment, then each man of the two remaining regiments would receive 50 cents; if the dollar were adjudged to the 2d regiment, then each man of the two remaining (1st and 3d) regiments would receive $\frac{1}{3}$ dollar; finally, if the dollar be adjudged to the third regiment, then each man of the remaining 1st and 2d regiment would receive only 25 cents. What is the number of men in each regiment?

Ans. The first consists of 780 men, the 2d of 1716, and the third of 2028.

68) It is required to determine three numbers from the following data: If the first be added to m times the others, then the sum $= a$; if the 2d be added to m' times the others, then the sum $= a'$; but if the third be added to m'' times the others, then the sum $= a''$. What numbers are they?

Ans. By making, in the course of the solution of this problem,

$$\frac{m}{m-1} + \frac{m'}{m'-1} + \frac{m''}{m''-1} = A,$$

$$\frac{a}{m-1} + \frac{a'}{m'-1} + \frac{a''}{m''-1} = B,$$

then the three numbers required may be expressed by

$$\frac{1}{m-1}\left(\frac{mB}{A-1} - a\right),\ \frac{1}{m'-1}\left(\frac{m'B}{A-1} - a'\right),$$

and

$$\frac{1}{m''-1}\left(\frac{m''B}{A-1} - a''\right)$$

and their sum is $= \frac{B}{A-1}$.

69) If, in the last problem, instead of three, there were four, five, six, &c, numbers to be determined, in the same manner, what would be their expressions?

Ans. Putting

$$\frac{m}{m-1} + \frac{m'}{m'-1} + \frac{m''}{m''-1} + \frac{m'''}{m'''-1} + \&c, = A,$$

$$\frac{a}{m-1} + \frac{a'}{m'-1} + \frac{a''}{m''-1} + \frac{a'''}{m'''-1} + \&c, = B,$$

we can express the numbers sought by the following formulæ:

$$\frac{1}{m-1}\left(\frac{mB}{A-1}-a\right),\ \frac{1}{m'-1}\left(\frac{m'B}{A-1}-a'\right),$$

$$\frac{1}{m''-1}\left(\frac{m''B}{A-1}-a''\right),\ \frac{1}{m'''-1}\left(\frac{m'''B}{A-1}-a'''\right),\&c,$$

and the sum of all these numbers is still

$$=\frac{B}{A-1}.$$

70) There are three numbers whose sum is 83; if from the first and 2d you subtract 7, the remainders are as 5 : 3; but if from the 2d and 3d you subtract 3, then their remainders are as 11 : 9. What numbers are they?

Ans. 37, 25, 21.

71) Required to find three numbers of the following properties: If 6 be added to the first and 2d, the sums are to one another as 2 : 3; if 5 be added to the first and 3d, the sums are as 7 : 11; but if 36 be subtracted from the 2d and 3d, then the remainders are as 6 : 7. What numbers are they?

Ans. 30, 48, 50.

72) A certain number consists of three digits, which are in an arithmetical progression. If this number be divided by the sum of its digits, (that is, without considering the value they have as tens and hundreds), the quotient is 48; but if 198 be subtracted from it, then we obtain for the remainder a number consisting of the same digits as the one sought, but in an inverted order. What number is this?

Ans. 432.

XVII. PROBLEMS FOR QUADRATIC EQUATIONS, WITH ONE AND MORE UNKNOWN QUANTITIES.

1) What number is it, whose half multiplied by its third part, gives 864?

Ans. 72.

2) What number is it, whose 7th and 8th part multiplied together, and the product divided by 3, gives the quotient $298\frac{2}{3}$?

Ans. 224.

3) It is required to find a number such, that if we first add to it 94, then subtract it from 94, and multiply the sum thus obtained, by the difference, the product may be 8512. What number is it?

Ans. 18.

4) What two numbers are they, whose product is 750, and quotient $3\frac{1}{3}$?

Ans. 50 and 15.

5) The product of two numbers is $= a$; their quotient $= b$. What formulæ give these two numbers?

Ans. $\sqrt{ab}$ and $\sqrt{\frac{a}{b}}$.

6) Find two numbers, the sum of whose squares $= 13001$, and the difference of their squares $= 1449$.

Ans. 85 and 76.

7) The sum of the squares of two numbers $= a$, the

difference of their squares $= b$. What formulæ will express them?

Ans. $\sqrt{\frac{a+b}{2}}$, $\sqrt{\frac{a-b}{2}}$.

8) What numbers are to one another as 3 to 4, and the sum of the squares of which $= 324900$?

Ans. 342, 456.

9) What two numbers are as m to n, and the sum of whose squares $= b$?

Ans. $\frac{m\sqrt{b}}{\sqrt{(m^2+n^2)}}$, $\frac{n\sqrt{b}}{\sqrt{(m^2+n^2)}}$.

10) What two numbers are to one another as m to n, and the difference of whose squares $= b$?

Ans. $\frac{m\sqrt{b}}{\sqrt{(m^2-n^2)}}$, $\frac{n\sqrt{b}}{\sqrt{(m^2-n^2)}}$.

11) A certain capital is out at 4 per cent; if we multiply the number of dollars in the capital, by the number of dollars in the interest for 5 months, we obtain \$117041$\frac{2}{3}$. What is the capital?

Ans. \$2650.

12) A person has three kinds of goods, which together cost \$230$\frac{5}{24}$. The pound of each article costs as many threepences ($\frac{1}{24}$ dollars) as there are pounds of that article; but he has one third more of the 2d kind than he has of the first, and $3\frac{1}{2}$ times as much of the 3d as he has of the 2d. How many pounds has he of each article?

Ans. Of the first 15, of the 2d 20, and of the 3d 70 pounds.

13) A person has on hand a small quantity of a certain

article. On my asking him how many lbs. there were of it, he answered: If I sold the lb. for $2\frac{3}{5}$ times as many three-pences as there are lbs., I should get by that means exactly as much above $\$6\frac{1}{2}\frac{1}{4}$, as I shall get less than this sum if I sell the lb. for half as many three-pences as there are lbs. How many lbs. had he?

Ans. 10 lbs.

14) I have a certain number in my thoughts; this I multiply by $2\frac{1}{3}$, add 7 to the product, multiply this sum by 8 times the number; I then divide by 14, and subtract from the quotient 4 times the number, and obtain 2352. What number is it?

Ans. 42.

15) Required to find three numbers, such, that the product of the first and 2d $= a$, the product of the first and 3d $= b$, and the sum of the squares of the 2d and 3d $= c$. How are these numbers expressed?

Ans. $\sqrt{\frac{a^2+b^2}{c}}$, $a\sqrt{\frac{c}{a^2+b^2}}$, $b\sqrt{\frac{c}{a^2+b^2}}$.

16) What three numbers are they, which, multiplied two and two, and each product divided by the third number, give the quotients a, b, c?

Ans. $\sqrt{ab}$, $\sqrt{ac}$, $\sqrt{bc}$.

17) What three numbers possess these properties, that the product of the first and 2d, the product of the 2d and 3d, and the product of the first and 3d, give respectively the numbers a, b, c?

Ans. $\sqrt{\frac{ac}{b}}$, $\sqrt{\frac{ab}{c}}$, $\sqrt{\frac{bc}{a}}$.

18) What five numbers possess these properties, that if each, beginning with the first, be multiplied by the one which follows it, but the last again by the first, the products are a, b, c, d, e?

Ans. $\sqrt{\frac{ace}{bd}}, \sqrt{\frac{abd}{ce}}, \sqrt{\frac{bce}{ad}}, \sqrt{\frac{acd}{be}}, \sqrt{\frac{bde}{ac}}$.

19) But, if instead of five, seven numbers are required, and the products are a, b, c, d, e, f, g; what numbers are they then?

Ans. $\sqrt{\frac{aceg}{bdf}}, \sqrt{\frac{abdf}{ceg}}, \sqrt{\frac{bceg}{adf}}, \sqrt{\frac{acdf}{beg}},$
$\sqrt{\frac{bdeg}{acf}}, \sqrt{\frac{acef}{bdg}}, \sqrt{\frac{bdfg}{ace}}$.

Similar formulæ may be found for every odd number of numbers sought, but only under certain conditions for an even number. Why? And what conditions must the numbers a, b, c, &c, be subject to, in order that it may be possible?

20) There are two numbers, one of which is greater than the other by 8, and whose product is 240. What numbers are they?

Ans. 12 and 20.

21) The sum of two numbers is $= a$, their product $= b$. What numbers are they?

Ans. $\frac{a + \sqrt{(a^2 - 4b)}}{2}, \frac{a - \sqrt{(a^2 - 4b)}}{2}$.

22) It is required to find a number whose square exceeds it by 306.

Ans. 18.

23) It is required to find a number such, that if we multi-

ply its 3d part by its 4th, and to the product add 5 times the number required, the sum exceeds the number 200 by as much as the number sought is less than 280?

Ans. 48.

24) A person being asked his age, answered, "my mother was 20 years old when I was born, and her age multiplied by mine, exceeds our united ages by 2500." What was his age?

Ans. 42.

25) A merchant has two kinds of goods, of different weight and prices. The weight of the first kind is to the weight of the 2d, as 4 : 3. The lb. of the first costs half as many three-pences ($\frac{1}{24}$ dollar) as it weighs in lbs.; the lb. of the 2d costs $\frac{1}{4}$ dollar less than a lb. of the first. The value of both kinds is $\$218\frac{1}{3}$. What is the weight of each?

Ans. The first 80 lbs., the 2d 60 lbs.

26) A merchant has three pieces of cloth, of which the 2d has 3, and the 3d 5 yards more than the first. Each yard of the first piece costs as many three-pences as it has yards; of the 2d, a yard costs $\frac{5}{12}$ dollar, and a yard of the 3d is $\frac{5}{6}$ dollar more than a yard of the first piece. The joint value of the three pieces is $\$397\frac{1}{12}$. How many yards are there in the first piece?

Ans. 50.

27) Determine the fortunes of three persons, A, B, C, from the following data: For every \$5 which A possesses, B has \$9, and C \$10. Farther, if we multiply A's money (expressed in dollars, and considered merely as a number) by B's, and B's money by C s, and add both products to the united fortunes of all three, we shall get 8832. How much had each?

Ans. A \$40, B \$72, C \$80.

28) A person buys some pieces of cloth, at equal prices, for \$60. Had he got 3 more pieces for the same sum, each piece would have cost him 1 dollar less. How many pieces did he buy?

Ans. 12.

29) A charitable person divides the sum of \$36, in equal shares, amongst the poor of a small town; but as 6 of those whom he thought of relieving, stood no longer in need of assistance, each of the remaining paupers had for his share six-pence ($\frac{1}{12}$ dollar) more than he otherwise would have had. How many paupers were there at first?

Ans. 54.

30) A person dies, leaving children, and a fortune of \$46800, which, by the will, is to be divided equally amongst them. It happens, however, that immediately after the death of the father, two of his children also die. If, in consequence of this, each remaining child receives \$1950 more than he or she was entitled to by the will, how many children were there?

Ans. 8 children.

31) Required to find a number, such, that if a given number c be divided by it, and also by one greater than it by a, the difference of the two quantities $= d$. What number is it?

$$\textit{Ans.}\ -\frac{a}{2} \pm \sqrt{\left(\frac{a^2}{4} + \frac{ac}{d}\right)}.$$

Are the three preceding problems included in this last?

32) 20 persons, men and women, together, spent \$48 at an inn; the men \$24, and the women the same sum. Now, on inspecting the bill, it is found that the men have to pay

one dollar each more than the women. How many men, therefore, were there in the company?

Ans. 8.

33) A person buys a horse, and pays a certain sum for it; he afterwards sells it again for $144, and gains exactly as much per cent, as the horse has cost him. How much did the horse cost?

Ans. $80.

34) A merchant sends for a piece of stuff, and pays a certain sum for it, besides 4 per cent for carriage. He sells it again for $390, and gains by the bargain as much per cent as the 12th part of the purchase-money amounted to. What did he buy it for?

Ans. $300.

35) Two drapers cut, each of them a certain number of yards, from a piece of cloth; one, however, 3 yards less than the other, and jointly receive for them $35. "At my own price," said the first to the other, "I should have received $24 for your cloth." "I must admit," answered the other, "that at my low price, I should have received for your cloth no more than $12½." How many yards did each sell?

Ans. The one 15, the other 18; or the one 5, and the other 8.

36) Two travellers, A and B, set out at the same time, from two different places, C and D; A, from C to D; and B, from D to C. On the way they met, and it then appears that A had already gone 30 miles more than B, and, according to the rate at which they travel, A calculates that he can reach the place D in 4 days, and that B can arrive at the place C in 9 days. What is the distance between C and D?

Ans. 150 miles.

37) In the preceding problem, let d be the distance which A had travelled more than B; a the time which A requires to finish the remainder of his journey, and b the time which B requires in order to finish his. What expression will give the distance between C and D?

Ans. $\frac{d(\sqrt{b}+\sqrt{a})}{\sqrt{b}-\sqrt{b}}$.

38) Two retailers jointly invested $500 in business, to which each contributed a certain sum; the one let his money remain 5 months, the other only 2, and each received back $450, capital and profit. How much did each advance?

Ans. One $200, the other $300.

39) Two persons jointly invested $2000 in business. One let his money remain 17 months, and received back, in capital and profit, $1710; the other allowed his money to remain 12 months, and received, in capital and gain, $1040. How much did each advance?

Ans. One $1200, the other $800.

40) What two numbers are they, whose sum is 41, and the sum of whose squares is 901?

Ans. 15 and 26.

41) The sum of two numbers is $= a$, the sum of their squares $= b$. What numbers are they?

Ans. $\frac{a+\sqrt{(2b-a^2)}}{2}$, and $\frac{a-\sqrt{(2b-a^2)}}{2}$.

42) What two numbers are they, whose difference is 8, and the sum of whose squares is 544.

Ans. 12 and 20.

43) What two numbers are they, whose product is 255, and the sum of whose squares is 514?

Ans. 15 and 17.

44) Divide the number 16 into two such parts, so that the products of the two parts added to their squares may be = 208.

Ans. Into 4 and 12.

45) Into what two parts must 39 be divided, so that the sum of their cubes may be = 17199?

Ans. Into 15 and 24.

46) A person being asked about his yearly income, answered, "My income is such, that if I add $1578 to it, and also subtract $142 from it, and extract the cube roots of the numbers thus obtained, the difference between the roots is 10." What was his income?

Ans. $150.

47) What number is it, which added to its square root gives 1332?

Ans. 1296.

48) What number exceeds its square root by $48\frac{3}{4}$?

Ans. $56\frac{1}{4}$.

49) There are two numbers, a and b, given; it is required to divide each of them into two such parts, that the one part of a is to the one part of b, as m to n; and that the product of the two other parts $= p$. How must these two numbers be divided?

Ans. Put $\frac{na + mb \pm \sqrt{(na - mb)^2 + 4mnp}}{2mn} = A$;

then the one part of a is $= mA$, and the one part of b is $= nA$.

50) Again, let it be required, as in the preceding problem, to divide the two numbers, a and b, so that the first parts may be to one another as m to n, but the sum of the squares of the two others $= s$. How must they, then, be divided?

Ans. Put $\dfrac{am + bn \pm \sqrt{(m^2 + n^2)\,s - (an - bm)^2}}{m^2 + n^2} = A$;

then the first part of a is $= m\,A$, and the first part of b is $= n\,A$.

51) Find two numbers, whose difference added to the difference of their squares, makes 150, and whose sum added to the sum of their squares, is 330. What numbers are they?

Ans. 9 and 15.

52) What numbers are they, whose sum, product, and difference of their squares, are all equal to one another?

Ans. $\dfrac{3 \pm \sqrt{5}}{2}$, $\dfrac{1 \pm \sqrt{5}}{2}$.

53) There are three numbers in continued proportion; if we add them, their sum is 126, but if we multiply them together, the product is 13824. What numbers are they?

Ans. 6, 24, and 96.

54) It is required to find a number, consisting of three digits, such, that the sum of the squares of the digits, without considering their position, may be $= 104$; but the square of the middle digit exceeds twice the product of the other two by 4; farther, if 594 be subtracted from the number sought, the three digits become inverted. What number is it?

Ans. 862.

* * *

We must not always infer, that the quantities required to be found are the unknown quantities (x, y, z, &c), *in the calculation; for this would in many cases lead us to higher equations than are necessary for the solution of the problem, and this we must try to avoid as much as possible. It is often better, first to seek some combination of the unknown quantities, perhaps their sum, difference, product, the sum or difference of their squares, &c, by which means, then, the unknown quantities themselves may finally be determined. As this is a very important point in Algebra, and one which cannot be too well observed, I shall now give a number of such problems.* (More of the same kind will occur farther on).

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55) Find two numbers, whose difference, multiplied by the difference of their squares, = 160; and whose sum, multiplied by the sum of their squares, gives the number 580.

Ans. The sum of the two numbers is 10, and their product 21; therefore the numbers themselves are 3 and 7.

56) Find two numbers such, that their sum and product together may be = 34, and the sum of their squares exceed the sum of the numbers themselves by 42. What numbers are they?

Ans. The sum of both numbers is 10, their product 24; consequently the numbers themselves are 4 and 6.

57) If, in order to make the foregoing problem more general, a be put instead of 34, and b instead of 42, by what formulæ will the numbers sought be expressed?

Ans. If we put $-1 \pm \sqrt{(4b + 8a + 1)} = 2A$,
and $2a + 1 \mp \sqrt{(4b + 8a + 1)} = 2B$,
then the two numbers sought are

$$\frac{A + \sqrt{(A^2 - 4B)}}{2}, \quad \frac{A - \sqrt{(A^2 - 4B)}}{2}.$$

58) What two numbers are they, whose sum $= a$, and the sum of whose 4th powers $= b$?

Ans. Call the difference of the two numbers sought $= d$, then $d = \sqrt{[-3a^2 \pm \sqrt{(8a^4 + 8b)}]}$; and the numbers themselves are $\frac{a + d}{2}, \frac{a - d}{2}$.

59) The sum of two numbers is $= a$, the sum of their 5th powers $= b$. What numbers are they?

Ans. The product p of both numbers is $=$

$$\tfrac{1}{2}\left[a^2 \pm \sqrt{\frac{a^5 + 4b}{5a}}\right];$$

consequently the numbers themselves are

$$\tfrac{1}{2}[a + \sqrt{(a^2 - 4p)}], \quad \tfrac{1}{2}[a - \sqrt{(a^2 - 4p)}].$$

60) The sum of two numbers is $= a$, their product multiplied by the sum of their squares is $= b$. What numbers are they?

Ans. Let the product of the two numbers $= p$; then

$$p = \tfrac{1}{4}[a^2 \pm \sqrt{(a^4 - 8b)}];$$

and the numbers themselves are

$$\tfrac{1}{2}[a + \sqrt{(a^2 - 4p)}], \quad \tfrac{1}{2}[a - \sqrt{(a^2 - 4p)}].$$

61) The sum of two numbers, added to the sum of their

squares $= a$; m times the sum of their squares, added to n times the product of the numbers, is $= b$. What numbers are they?

Ans. The sum s, and the product p, of both numbers are given by the equations

$$ns^2 + (n - 2m)s = 2b + (n - 2m)a,$$

$$2p = s^2 + s - a.$$

Having determined from these s and p; the numbers themselves may be found by solving the equation $x^2 - sx + p = 0$. Each number, therefore, has 4 values.

62) In a geometrical proportion, the sum of the means $= a$, the sum of the extremes $= b$, and the sum of the squares of all four terms $= c$. What is the proportion?

Ans. The product of the two means, consequently, also, the product of the two extremes $= \dfrac{a^2 + b^2 - c}{4}$, therefore, the required proportion is

$$\tfrac{1}{2}[b - \sqrt{(c - a^2)}] : \tfrac{1}{2}[a - \sqrt{(c - b^2)}]$$

$$= \tfrac{1}{2}[a + \sqrt{(c - b^2)}] : \tfrac{1}{2}[b + \sqrt{(c - a^2)}].$$

63) The difference between the two mean terms of a geometrical proportion $= a$; the difference between the two extremes $= b$, and the sum of the squares of all four terms $= c$. What is the proportion?

Ans. The product of the two extremes, and therefore, also, of the two means is $\dfrac{c - a^2 - b^2}{4}$; consequently the proportion is

$$\tfrac{1}{2}[-b + \sqrt{(c - a^2)}] : \tfrac{1}{2}[-a + \sqrt{(c - b^2)}]$$

$$= \tfrac{1}{2}[+a + \sqrt{(c - b^2)}] : \tfrac{1}{2}[+b + \sqrt{(c - a^2)}].$$

64) In a geometrical proportion, the product of the two extremes, or means, is $= a$, the sum of all four terms $= b$; and the sum of their squares $= c$. Find the proportion.

Ans. For shortness' sake, let $\pm\sqrt{(8a + 2c - b^2)} = A$; then $\frac{b - A}{2}$ is the sum of the two means, and $\frac{b + A}{2}$, the sum of the two extremes; consequently, the proportion sought is

$$\tfrac{1}{4}[b + A - \sqrt{(2c - 8a + 2bA)}] : \tfrac{1}{4}[b - A - \sqrt{(2c - 8a - 2bA)}]$$
$$= \tfrac{1}{4}[b - A + \sqrt{(2c - 8a - 2bA)}] : \tfrac{1}{4}[b + A + \sqrt{(2c - 8a + 2bA)}]$$

65) The product of the two means, or extreme terms of a geometrical proportion $= a$, the difference between the sum of the extremes, and the sum of the means $= b$, and the sum of the squares of all four terms $= c$. What is the proportion?

Ans. Again, let $\pm\sqrt{(8a + 2c - b^2)} = A$; then $\frac{A - b}{2}$ is the sum of the means, and $\frac{A + b}{2}$ the sum of the extremes; the proportion sought is

$$\tfrac{1}{4}[A + b - \sqrt{(2c - 8a + 2bA)}] : \tfrac{1}{4}[A - b - \sqrt{(2c - 8a - 2bA)}]$$
$$= \tfrac{1}{4}[A - b + \sqrt{(2c - 8a - 2bA)}] : \tfrac{1}{4}[A + b + \sqrt{(2c - 8a + 2bA)}].$$

When $a = 18$, $b = 2$, $c = 130$; then $2 : 3 = 6 : 9$.

When $a = 270$, $b = 20$, $c = 3922$; then $5 : 9 = 30 : 54$.

66) In a geometrical proportion the product of the two mean or extreme terms $= a$, the sum of all the terms $= b$, and the difference between the sum of the squares of the extremes, and the sum of the squares of the means $= c$. What is the proportion?

Ans. The sum of the two mean terms is $= \frac{b^2 - c}{2b}$, the sum of the two extremes $= \frac{b^2 + c}{2b}$; consequently, the proportion sought is

$$\frac{b^2 + c - \sqrt{[(b^2 + c)^2 - 16ab^2]}}{4b} : \frac{b^2 - c - \sqrt{[(b^2 - c)^2 - 16ab^2]}}{4b} =$$

$$\frac{b^2 - c + \sqrt{[(b^2 - c)^2 - 16ab^2]}}{4b} : \frac{b^2 + c + \sqrt{[(b^2 + c)^2 - 16ab^2]}}{4b}.$$

67) Required to find three numbers in a continual proportion, whose sum $= a$, and the sum of whose squares $= b$.

Ans. The middle term of the required proportion is $= \frac{a^2 - b}{2a}$; the two extremes consequently are

$$\frac{a^2 + b - \sqrt{(3b - a^2)(3a^2 - b)}}{4a},$$

$$\frac{a^2 + b^2 + \sqrt{(3b - a^2)(3a^2 - b)}}{4a}.$$

Between what limits must the value of b fall, in order that this problem may give possible results?

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68) In a continued proportion, the sum of the three terms is $= a$, and the remainder, which we obtain when we subtract the square of the mean from the sum of the squares of the extremes, $= b$. What proportion is it?

Ans. Call the middle term g; then

$$g = \frac{-a \pm \sqrt{(3a^2 - 2b)}}{2};$$ whence the extremes

$$\tfrac{1}{2}[a - g \pm \sqrt{(a^2 - 2ag - 3g^2)}],\ \tfrac{1}{2}[a - g \mp \sqrt{(a^2 - 2ag - 3g^2)}].$$

69) In a geometrical progression of four terms, the sum of all the terms $= a$, the sum of their squares $= b$. What progression is it?

Ans. Let s denote half the sum, and d half the difference of the two mean terms; then

$$s = \frac{-b \pm \sqrt{[b^2 + 2a^2(a^2 - b)]}}{4a}; \text{ and } d = s\sqrt{\frac{a-4s}{a+4s}};$$

whence we obtain the two terms $s - d$, $s + d$, and the two extremes $\frac{(s-d)^2}{s+d}$, $\frac{(s+d)^2}{s-d}$.

70) In a geometrical proportion of four terms, the following things are given: The difference between the sum of the extremes and the sum of the means $= a$, and also the difference between the sum of the squares of the two extremes, and the sum of the squares of the two mean terms $= b$. What proportion is it?

Ans. Let s stand for half the sum of the two mean terms, d for half the difference; then we have

$$s = \frac{b - a^2}{4a}, \quad d = \frac{b - a^2}{4\sqrt{(2b - a^2)}};$$

whence we obtain the two means $s - d$, $s + d$, and the two extremes $\frac{(s-d)^2}{s+d}$, $\frac{(s+d)^2}{s-d}$.

71) In a geometrical progression of four terms, we have the difference between the sum of the 2d and 4th term, and the sum of the first and 3d term $= a$, and the sum of the squares of all four terms $= b$. What progression is it?

Ans. Let half the difference between the mean terms be

$= d$, their half sum $= s$; then

$$d = \frac{b \pm \sqrt{[b^2 + 2a^2(a^2 - b)]}}{4a}, \quad s = d\sqrt{\frac{a + 4d}{a - 4d}};$$

whence we obtain, as in the foregoing problems, the means and extremes.

72) In a geometrical progression of four terms, we have the sum of all the terms $= a$, the difference between the sum of the squares of the extremes, and the sum of the squares of the means $= b$. Find the progression?

Ans Half the sum of the means, $s = \dfrac{a^2 - b}{4a}$; half the difference $d = \pm s\sqrt{\dfrac{b}{8as + b}}$; whence we obtain the rest, as in the three preceding problems.

73) In a geometrical progression of four terms, the sum of the two extremes $= a$, the sum of the two means $= b$. What is the progression?

Ans. Let the exponent of the progression be e; then

$$e = \frac{a + b \pm \sqrt{(a - b)(a + 3b)}}{2b},$$

and the first term $\dfrac{a}{e^3 + 1} = \dfrac{b}{e^2 + e}$.

74) In a geometrical proportion, we have the sum of the means $= a$, the sum of the extremes $= b$, the sum of the cubes of all four terms $= c$. What is the proportion?

Ans. Let the product of the means or extremes be p; then $p = \dfrac{a^3 + b^3 - c}{3(a + b)}$, and the required proportion

$$\tfrac{1}{2}[b - \sqrt{(b^2 - 4p)}] : \tfrac{1}{2}[a - \sqrt{(a^2 - 4p)}]$$
$$= \tfrac{1}{2}[a + \sqrt{(a^2 - 4p)}] : \tfrac{1}{2}[b + \sqrt{(b^2 - 4p)}].$$

75) In a geometrical proportion, the sum of all the terms $= a$, the sum of their squares $= b$, the sum of the cubes of all four terms $= c$. Which is it?

Ans. The product of the means or extremes is

$$p = \frac{a^3 - 3ab + 2c}{6a},$$

the difference between the sum of the extremes and the sum of the means is

$$d = \pm\sqrt{\frac{a^3 - 6ab + 8c}{3a}};$$

therefore, the sum of the extremes $= \frac{a + d}{2}$, the sum of the means $= \frac{a - d}{2}$; whence the proportion is

$$\begin{aligned} &\tfrac{1}{4}[a + d - \sqrt{[(a + d)^2 - 16p]}] \\ &: \tfrac{1}{4}[a - d - \sqrt{[(a - d)^2 - 16p]}] \\ &= \tfrac{1}{4}[a - d + \sqrt{[(a - d)^2 - 16p]}] \\ &: \tfrac{1}{4}[a + d + \sqrt{[(a + d)^2 - 16p]}]. \end{aligned}$$

76) In a geometrical proportion the difference between the sum of the means and the sum of the extremes $= a$, the difference between the sum of the squares of the means and the sum of the squares of the extremes $= b$; farther, the difference between the sum of the cubes of the extremes and the sum of the cubes of the means $= c$. What proportion is it?

Ans. The sum of all the terms is $= \frac{b}{a}$, whence the sum of the extremes, $s = \frac{b + a^2}{2a}$, the sum of the means

$s' = \dfrac{b - a^2}{2a}$, the product of the extremes or means

$$p = \frac{a^4 + 3b^2 - 4ac}{12a^2}.$$

Now, that s, s' and p, are known, the proportion sought is

$$\tfrac{1}{2}[s - \sqrt{(s^2 - 4p)}] : \tfrac{1}{2}[s' - \sqrt{(s'^2 - 4p)}]$$
$$= \tfrac{1}{2}[s' + \sqrt{(s'^2 - 4p)}] : \tfrac{1}{2}[s + \sqrt{(s^2 - 4p)}].$$

77) In a geometrical proportion the product of the two extremes or means $= a$, the sum of all the terms $= b$, the sum of their cubes $= c$. What proportion is it?

Ans. For shortness' sake let

$$\pm\sqrt{\frac{4c + 12ab - b^3}{3b}} = \mathcal{A},$$

then $\dfrac{b + \mathcal{A}}{2}$, is the sum of the extremes, and $\dfrac{b - \mathcal{A}}{2}$, the sum of the means; whence the proportion sought

$$\tfrac{1}{4}[b + \mathcal{A} - \sqrt{[(b + \mathcal{A})^2 - 16a]}]$$
$$: \tfrac{1}{4}[b - \mathcal{A} - \sqrt{[(b - \mathcal{A})^2 - 16a]}]$$
$$= \tfrac{1}{4}[b - \mathcal{A} + \sqrt{[(b - \mathcal{A})^2 - 16a]}]$$
$$: \tfrac{1}{4}[b + \mathcal{A} + \sqrt{[(b + \mathcal{A})^2 - 16a]}].$$

78) In a geometrical proportion the sum of the two extremes $= a$, the sum of the two means $= b$, and the sum of the cubes of all four terms $= c$. What is the proportion?

Ans. Let $\sqrt{\dfrac{4c - a^3 - 4b^3 + 3a^2b}{3(a + b)}} = \mathcal{A}$,

$$\sqrt{\frac{4c - 4a^3 - b^3 + 3ab^2}{3(a + b)}} = B;$$

then $\dfrac{a - \mathcal{A}}{2} : \dfrac{b - B}{2} = \dfrac{b + B}{2} : \dfrac{a + \mathcal{A}}{2}$, is the proportion sought.

79) How are the following equations solved, in which x', x'' are the quantities sought?

$$(x'+x'')(1+x'x''+x'^2x''+x'x''^2+x'^2x''^2)+x'x''=a$$
$$x'x''(x'+x'')(x'+x''+x'x'')(x'+x''+x'x''+x'^2x''+x'x'')=b.$$

Ans. If, in these equations, we make successively the substitutions $x'+x''=y'$, $x'x''=y''$; $y'+y''=z'$, $y'y''=z''$; $z'+z''=w'$, $z'z''=w''$, we obtain at length $w'+w''=a$, $w'+w''=b$. The quantities sought, viz. x', x'', are, therefore, determined by the four following quadrâtic equations:

$$\begin{aligned} w^2-aw+b&=0\\ z^2-w'z+w''&=0\\ y^2-z'y+z''&=0\\ x^2-y'x+y''&=0\end{aligned}$$

The first of these equations determines w' w''; the 2d z', z''; the 3d y', y''; and finally, the 4th x', x''. We thus successively obtain

$$w'=\frac{a\pm\sqrt{(a^2-4b)}}{2},\quad w''=\frac{a\mp\sqrt{(a^2-4b)}}{2},$$

$$z'=\frac{w'\pm\sqrt{(w'^2-4w'')}}{2},\quad z''=\frac{w'\mp\sqrt{(w'^2-4w'')}}{2},$$

$$y'=\frac{z'\pm\sqrt{(z'^2-4z'')}}{2},\quad y''=\frac{z'\mp\sqrt{(z'^2-4z'')}}{2},$$

$$x'=\frac{y'\pm\sqrt{(y'^2-4y'')}}{2},\quad x''=\frac{y'\mp\sqrt{(y'^2-4y'')}}{2};$$

and there are consequently, for x', as well as for x'', 16 different values. If we he had solved the first two equations by the common method, we should, after a laborious elimination, have obtained an equation of the 16th degree.

XVIII. PROBLEMS FOR EQUATIONS OF HIGHER DEGREES.

1) Which number is it, whose third part multiplied by its square, gives 1944?

Ans. 18.

2) What number is it, whose half, third, and fourth, multiplied together, and the product increased by 32, gives 4640?

Ans. 48.

3) Find a number such, that its fourth power divided by the 8th part of the number, and 167 subtracted from the quotient, may give the remainder 12000. What number is it?

Ans. $11\frac{1}{2}$.

4) Some merchants engage in business; each gives to it a thousand times as many dollars as there are partners. They gain, in this business, $2560; and it is found that they have gained exactly half their own number per cent. How many merchants are there?

Ans. 8.

5) A capitalist puts out $10000 at interest, and adds the interest yearly to the capital. At the end of the third year he finds his capital increased to $\$11576\frac{1}{4}$. How many per cent did it bear?

Ans. 5 per cent.

6) It is required to find three numbers of the following properties: If the square of the first be multiplied by the second, the product = 112; if the square of the second be multiplied by the third, we get 588; but if we multiply the

square of the third by the first, we obtain 576. What numbers are they?

Ans. 4, 7, 12.

7) Find three numbers, such, that the square of the first multiplied by the 2d $= a$, the square of the 2d multiplied by the 3d $= b$, and the square of the 3d multiplied by the first $= c$. What numbers are they?

$$Ans.\ \sqrt[9]{\frac{a^4c}{b^2}},\ \sqrt[9]{\frac{b^4a}{c^2}},\ \sqrt[9]{\frac{c^4b}{a^2}}.$$

8) But when four numbers are sought, and it is required that the products, a, b, c, d, should be successively obtained when the square of each number is multiplied by its succeeding one, and the square of the last again by the first; what numbers will they then be?

$$Ans.\ \sqrt[15]{\frac{a^8c^2}{b^4d}},\ \sqrt[15]{\frac{b^8d^2}{c^4a}},\ \sqrt[15]{\frac{c^8a^2}{d^4b}},\ \sqrt[15]{\frac{d^8b^2}{a^4c}}.$$

(Similar formulæ may be found for five, six, and more numbers. The problem admits also of being made more general; all which is left to the learner's own consideration.)

9) A person bottled off a certain quantity of wine from a full wine cask, which contains 81 gallons. When he had, afterwards, filled it again with water, he drew-off, again, from this mixture, exactly as much as before; this he did four times successively, until there were no more than 16 gallons of pure wine in the cask; the remainder being water. How many gallons did he draw off each time?

Ans. 27.

10) There are two numbers whose difference is 4, and,

moreover, are such, that their product multiplied by their sum, gives 1386. What numbers are they?

Ans. 7 and 11.

11) A person buys a silver vessel, which weighs exactly as many pounds as there are ounces of pure silver in a lb. He pays $120 for this vessel; viz. for each ounce of pure silver which it contains, $\frac{1}{3}$ dollar more, than the vessel would cost if he paid $\frac{1}{24}$ dollar for each pound of its weight. How much does it weigh?

Ans. 12 lbs.

12) Some officers were in a field with a detachment, partly infantry and partly cavalry. Each officer had under his command three times as many cavalry, and seven times as many foot as there are officers. Each cavalry soldier has 2, and each foot soldier 22 cartridges more than there are officers. They had altogether 15360 cartridges. How many officers were there?

Ans. 8.

13) A person was asked how much he had expended that day;—"To day," he answered, "I have spent $4 more, and yesterday twice as much as I did the day before yesterday; if I multiply together the sums which I expended in dollars during these three days, and add 756 to the product, I obtain exactly 134 times as much I have expended to day." How much, therefore, is it?

Ans. 6 or 9 dollars.

14) Some merchants jointly form a certain capital, in such a way, that each contributes 10 times as many dollars as they are in number; they trade with this capital, and gain as many dollars per cent as exceed the number of merchants

by 8. Their profit amounts to \$288. How many were there of them?

Ans. 12.

15) Some merchants collect a capital of \$8240. To this each contributes forty times as many dollars as there are of them. With this whole sum they gain as many dollars per cent as there are persons. They then divide the profit, and each takes ten times as many dollars as there are persons; but after this there remain \$244 over. How many merchants were there?

Ans. Either 7, 8, or 10.

16) Four persons, A, B, C, D, have each of them a certain number of dollars in their possession; B \$1 more than A, C \$1 more than B, and D \$1 more than C. If we multiply the four sums by one another, and consider the product as so many dollars, we shall obtain \$1168 more than we should by cubing D's. How much has each?

Ans. A \$5, B \$6, C \$7, D \$8.

17) A person has a certain number of workmen, who are three times as many in number as each receives three-pences ($\frac{1}{24}$ dollars) daily. They jointly work 100 days less than their daily hire amounts to in three-pences, and for this time they receive, jointly, \$2500. How many workmen were there? and how many days did they work?

Ans. 30 workmen, and 200 days.

18) I have two numbers, whose sum is 63. If the greater be divided by the less, this quotient multiplied by the greater, and $20\frac{1}{4}$ added to the product, there arises a cubic number, whose root is 1 less than the 7th part of the greater number. What are the numbers?

Ans. 35, 28.

19) A reservoir is supplied with water by 4 pipes, and can be filled by them in $115\frac{1}{5}$ minutes. But if the reservoir be filled by each pipe singly, then the 2d will require 4, the 3d 8, and the 4th 12 hours more than the first. In what time will it be filled by the first?

Ans. In 4 hours.

20) Three numbers are given by the following data: The sum of the 1st and 2d $= a$; the sum of the squares of the 2d and 3d $= b$; and the sum of the cubes of the 1st and 3d $= c$. By what equation will these three numbers be determined?

Ans. Let x denote the 1st of the three numbers sought, then $[b - (a - x)^2]^3 = (c - x^3)^2$, is the equation which we have to solve; for if x is determined from it, the other numbers are easily found.—A problem may be considered as solved, as soon as we have reduced it to the most simple equation it admits of, although we are not always able to solve the equation itself, completely.

21) We know that the sum of two numbers $= a$, that the sum of their 6th powers $= b$. How are these numbers found?

Ans. The product p of the two numbers is expressed by the equation $2p^3 - 9a^2p^2 + 6a^4p - a^6 = 0$.

22) The sum of two numbers $= a$, the sum of their 7th powers $= b$. How are these numbers found?

Ans. The product p of the two numbers is expressed by the equation $7ap^3 - 14a^3p^2 + 7a^5p - a^7 + b = 0$. In general, the two equations $x + y = a$, $x^{2n} + y^{2n} = b$, or $x^{2n+1} + y^{2n+1} = b$, always lead to an equation of the nth degree for p, whose law is easily discovered.

23) m times the difference of two numbers added to n times their product, gives a; their difference multiplied by the sum of their squares, gives b. What numbers are they?

Ans. Their difference $= y$, and their product $= z$, are expressed by the equations

$$ny^3 - 2my^2 + 2ay - nb = 0, \quad nz = a - my.$$

Having found y and z from these, we can determine the quantities themselves by the solution of a quadratic equation. The two quantities are properly given by an equation of the 6th degree, which may however, (as we have shown), be reduced to equations of the 3d and 2d degree.

24) The sum of three numbers $= a$, the sum of their products combined two and two $= b$, the product of all three $= c$. How are they expressed?

Ans. The equation $x^3 - ax^2 + bx - c = 0$, gives, by its three roots, the three numbers sought.

25) The sum of three numbers $= a$, the sum of their products two and two $= b$, and the sum of the six products which we obtain by multiplying each number by the square of one of the other two $= c$. How are these numbers expressed?

Ans. The product of the three numbers is $= \frac{ab - c}{3}$; therefore, $x^3 - ax^2 + bx - \frac{ab - c}{3} = 0$, the equation which gives all three numbers by its three roots.

26) The sum of three numbers $= a$, the sum of their squares $= b$, the sum of their cubes $= c$. How are these numbers found?

Ans. The sum of the products of these numbers, taken two and two, $= \frac{a^2 - b}{2}$, and the product of all three $= \frac{2c + a^3 - 3ab}{6}$; consequently the equation $x^3 - ax^2 + \frac{a^2 - b}{2}x - \frac{2c + a^3 - 3ab}{6} = 0$, gives, by its three roots, the numbers sought.

27) The sum of the squares of two numbers $= a$, the sum of their cubes $= b$. What numbers are they?

Ans. Let s denote the sum of the two numbers; then $s^3 - 3as + 2b = 0$, is the equation by which s is found, which, being once determined, the numbers themselves may be obtained by the equation

$$x^2 - sx + \frac{s^2 - a}{2} = 0.$$

28) The sum of the products of three numbers taken two and two is $= a$, the sum of their squares $= b$, the product of all three $= c$. Which are they?

Ans. The sum of these three numbers $= \pm \sqrt{(2a + b)}$; consequently, the equation whose roots give the number sought is $x^3 \mp x^2\sqrt{(2a + b)} + ax - c = 0$.

29) The sum of the products of three numbers taken two and two $= a$, the sum of the squares $= b$, the sum of their cubes $= c$. How are these numbers found?

Ans. The sum of the three numbers $= \pm \sqrt{(2a + b)}$; the product of all three

$$= \tfrac{1}{3}[c \pm (a - b)\sqrt{(2a + b)}];$$

consequently, the following equation determines the numbers themselves:

$$x^3 \mp x^2\sqrt{(2a+b)} + ax - \tfrac{1}{3}[c \pm (a-b)\sqrt{(2a+b)}] = 0.$$

30) The sum of the products of three numbers taken two and two is $= a$, the sum of their squares $= b$, the sum of the six products arising from multiplying each number by the square of another $= c$. How are they found?

Ans. The sum of the three numbers is $= \pm\sqrt{(2a+b)}$, and their product $= \tfrac{1}{3}[\pm a\sqrt{(2a+b)} - c]$; they are, consequently, given by the following equation:

$$x^3 \pm x^2\sqrt{(2a+b)} + ax - \tfrac{1}{3}[\pm a\sqrt{(2a+b)} - c] = 0.$$

31) Find three numbers in continued proportion, whose sum $= a$, and the sum of their cubes $= b$.

Ans. Let y denote the mean term of the proportion; then $3y^3 - 3a^2y + a^3 - b = 0$, is the equation by which it may be determined. Let w be a root of this equation; then

$$x^2 - (a-w)x - \frac{b - w^3}{3(a-w)} + \frac{(a-w)^2}{3} = 0,$$

is the equation whose roots are the two extreme terms of the proportion. When $a = 21$, $b = 1971$, we have

$$y^3 - 441y + 2430 = 0.$$

The three roots of this equation are

$$6, \quad -3 + \sqrt{414}, \quad -3 - \sqrt{414}.$$

Now, if we assume $w = 6$, then $x^2 - 15x + 36 = 0$, is the equation, whose two roots, 3 and 12, give the two extremes. The proportion is

$$3 : 6 = 6 : 12.$$

32) In a progression of four terms, we know that the difference between the two extremes is $= a$, and the sum of the two means $= b$; what progression is it?

Ans. If the exponent of the progression be denoted by y, then $by^3 - ay^2 - ay - b = 0$, is the equation by which it may be determined. Having found this, then the first term

$$= \frac{b}{y^2 + y} = \frac{a}{y^3 - 1}.$$

33) But what will the progression be, when the sum of the two means $= a$, and the sum of the squares of the two extremes $= b$?

Ans. Calling the difference between the two means $= d$, and $d^2 = y$; then

$$y^3 + (15a^2 - 2b)\,y^2 + (15a^4 + 4a^2b)\,y + a^4(a^2 - 2b) = 0,$$

is the equation which gives y. If y be determined from this, and consequently also d, we then obtain the exponent of the progression $= \frac{a + d}{a - d}$, and the first term $= \frac{(a - d)^2}{2(a + d)}$.

* * *

If we know previously, either from the nature of the problem which leads to a given equation, or from any other cause, something of the relation of the roots of this equation, then, with few exceptions, we can reduce it to a lower degree. The following problem may serve as an illustration:

34) Let $2m$ quantities be given by the equation

$$x^{2m} + ax^{2m-1} + bx^{2m-2} + cx^{2m-3} + \ldots\ldots kx + l = 0.$$

We know already, that when these quantities are combined two and two, the sums are the same. How, then, does this equation admit of being solved by another of a lower degree?

Ans. The sum of every two roots of this equation $= -\frac{a}{m}$.

By arranging it according to the powers of $x^2 + \frac{ax}{m}$, and then substituting y for this expression, we obtain an equation of the mth degree for y. If from this equation we can find y, we can get x from the equation $x^2 + \frac{ax}{m} = y$.

The demonstration of this depends upon the decomposition of every equation into single factors, and may be easily derived from it. (This is left to the student's consideration.)

To the same class belongs the equation

$$x^4 - 10x^3 + 18x^2 + 35x - 12 = 0.$$

Make $x^2 - 5x = y$, and give the equation the form

$$(x^2 - 5x)^2 - 7(x^2 - 5x) - 12 = 0.$$

Then the two equations

$$y^2 - 7y - 12 = 0, \quad x^2 - 5x - y = 0,$$

give the four required values of x, viz.

$$x = \frac{5 \pm \sqrt{(39 + 2\sqrt{97})}}{2}, \quad x = \frac{5 \pm \sqrt{(39 - 2\sqrt{97})}}{2}.$$

Supposing we knew beforehand that the equation

$$x^6 - 12x^5 + 42x^4 - 16x^3 - 79x^2 - 68x - 18 = 0,$$

has 3 pairs of roots, whose sums are equal; then we put $x^2 - 4x = y$, and give to the given equation the form

$$(x^2 - 4x)^3 - 6(x^2 - 4x^2)^2 + 17(x^2 - 4x) - 18 = 0.$$

Then, the two equations

$$y^3 - 6y^2 + 17y - 18 = 0, \quad x^2 - 4x = y,$$

give the six roots sought.

The three values of y are 2, $2+\sqrt{-5}$, $2-\sqrt{-5}$; hence, those of x are,

$$2 \pm \sqrt{6},\quad 2 \pm \sqrt{(6+\sqrt{-5})},\quad 2 \pm \sqrt{(6-\sqrt{-5})}.$$

With the help of indeterminate coefficients, it will always be easy to give to the equation the desired form.

XIX. INDETERMINATE, OR DIOPHANTIC PROBLEMS.

If, from the conditions of a problem, we cannot obtain as many equations as there are unknown quantities, then it belongs to the class of indeterminate problems. Let there be m equations of the form

$$ax + a'x' + a''x'' + a'''x''' + \&c, = k,$$

between the M unknown quantities x, x', x'', &c, and let $M > m$; then, by eliminating $m-1$ of these unknown quantities, we shall obtain an equation of the same form, in which there will only be $M-m+1$ of these unknown quantities. In order, then, to solve such an equation, it is only necessary to express one of the unknown quantities by all the rest, and then to substitute for the others arbitrary numerical values. If, for instance, we had the equation

$$3x + 5x' - 7x'' = 17,$$

we should find $x'' = \dfrac{3x + 5x' - 17}{7}$, where we can assume for x and x', arbitrary values, and thereby determine x''.

But there are usually certain other conditions added, which cannot be properly expressed by equations, by which means the solution becomes much more difficult. As, for instance, when it is required that x, x', x'', &c, shall all be integers; and when, besides, it is required that they shall all be positive, &c. In this case we can always assume, that the coefficient, k, a, a', a'', &c, are all whole numbers; for if that is not the case, we may get rid of the fractions by successive multiplication.

The equation $ax + a'x' = 1$, when it is made a condition that x and x' shall be integers, can be solved, either by the method given in most elementary books, or by continued fractions. The latter method depends more upon the rules VI, VII, page 142, and is much easier than the former. Having, by any method, found $x = p$, $x' = q$, so that $ap + a'q = 1$, we can, then, generally, put $x = p + a'n$, $x' = q - an$, and then assume for n, any arbitrary whole, positive, or negative number.

The equation $ax + a'x' = k$, then, gives $x = kp + a'n$, $x' = kq - an$. The equation

$$ax + a'x' + a''x'' + a'''x''' + \&c, = k,$$

consequently gives

$$x = (k - a''x'' - a'''x''' - \&c)\, p + a'n,$$

$$x' = (k - a''x'' - a'''x''' - \&c)\, q - an,$$

where for n, x''', x'', &c, all possible whole, positive, or negative numbers may be assumed.

The above m equations with M unknown quantities are, therefore, reducible to the equation $ax + a'x' = 1$, and this is always possible, unless a and a' have a common divisor. Among the coefficients, a, a', a'', a''', &c, when the equation

$$ax + a'x' + a''x'' + \&c, = k,$$

is possible, there must at least be two, which have no common divisor.

* * *

EXAMPLES.

1) What numbers divided by 3 leave the remainder 1, and divided by 5, the remainder 2?

Ans. 7, 22, 37, 52, 67, 82, &c, and, in general, all numbers of the form $15n + 7$.

2) What numbers divided by 8 leave 5, and divided by 11, the remainder 4?

Ans. 37, 125, 213, 301, 389, &c, and, in general, all numbers of the form $88n + 37$.

3) What numbers are divisible by 9, and when divided by 14, leave 8 for a remainder?

Ans. 36, 162, 288, 414, 540, &c, and, in general, all numbers of the form $126n + 36$.

4) A country woman brings a number of eggs to market, more than 100, but less than 200. She is undetermined whether to sell them in fifteens or by the dozen, for in the first case she would have 4 over, in the 2d 10. How many eggs had she?

Ans. 154.

5) A boy plays with nuts, the number of which lies between 1 and 400, and wishes to make small heaps of them. If he put 13 in each heap, he has 9 over; but if he put 17 in each, he has 14 over. How many nuts are there?

Ans. 269.

Do the last two problems really belong to the class of indeterminate problems?

6) What numbers divided by 3, 7, and 10, leave the remainders 2, 3, and 9, respectively.

Ans. 59, 269, 479, 689, 899, and so on; in general, all numbers of the form $210n + 59$.

7) What numbers divided by 6, 12, and 15, leave the remainders 1, 1, 10?

Ans. 25, 85, 145, 205, 265, and so on; in general all numbers of the form $60n + 25$.

8) What numbers divided by 5, 6, 7, 8, leave the remainders 3, 1, 0, 5, respectively?

Ans. 133, 973, 1813, 2653, 3493, &c; and, in general, all numbers of the form $840n + 133$.

9) What numbers divided by 4, 6, 9, 15, leave the remainder 3, for the first three divisors, and for the fourth, the remainder 12?

Ans. 147, 327, 507, 687, 867, and so on; and, in general, all numbers of the form $180n + 147$.

10) A general was asked how strong his regiment was. He answered, "My regiment is not 2000 men strong. I can, however, draw it up 5, 6, and 7 deep, without having one over; but if I wished to make it 11 and 13 deep, in the first case I should have 9 men over, and in the 2d, 8 too few." How many men did the regiment contain?

Ans. 1890 men.

11) A captain wished to march out his company, which was between 100 and 200 men strong. If he march them 2, 4, 8, and 10 men deep, he has 1 over each time; but if he march them 6 or 12 deep, then he has 5 over each time. What number of men did his company contain?

Ans. 161.

12) A fortunate gamester wanted to count the guineas he had won twice successively; the first time having told them by threes, he had 2 over; the 2d time he counted by fives, he had 1 over. After this he played again, when he lost 6 guineas, and then reckoned the money he had left by 7 and 11 at a time, and found that he had 3 over each time. How many guineas did he win the first time he played?

Ans. 86, or 1241, or 2396, and so on.

13) Required to find two numbers such, that if the first be multiplied by 17, and the second by 26, the first product is 7 greater than the second. What numbers are they?

Ans. 5 and 3, or 31 and 20, or 57 and 37, &c.

14) In a foundery, there were two kinds of cannon cast. Each cannon of the first sort weighs 16 cwt., and of the 2d kind 25 cwt.; and yet, for the 2d kind there were used 1 cwt. of metal less than for the first. How many cannons were there of each kind?

Ans. Of the first 11, and of the 2d 7, or of the first kind 36, and of the 2d 23; and so on.

15) Required to find three numbers such, that if the first be multiplied by 7, the 2d by 9, and the 3d by 11, the first product may be 1 less than the 2d, and 2 greater than the third?

Ans. 5, 4, 3; or 104, 81, 66; or 203, 158, 129, &c.

16) A number of men, women, and children, form a party of pleasure. Each man spends $\frac{19}{24}$, each woman $\frac{5}{12}$, and each child $\frac{1}{3}$ of a dollar. The men jointly spend $\frac{7}{24}$ of a dollar more than the women, and $\frac{5}{8}$ of a dollar more than the

children. How many men, women, and children were there?

Ans. 13 men, 24 women, and 29 children; or
53 men, 100 women, and 124 children, &c.

17) Divide 142 into two such parts, that the one may be divisible by 9, and the other by 14. What parts are they?

Ans. 72 and 70.

18) Divide the number 1591 into two such parts, that the one may be divisible by 23, and the other by 34. What parts are they?

Ans. 1081 and 510, or 299 and 1292.

19) Into what two parts must you divide the number 4890, that the first part divided by 37, may leave the remainder 3, and the 2d divided by 54, the remainder 6?

Ans. 780 and 4110, or in 2778 and 2112, or in
4776 and 114.

20) A bailiff buys horses and oxen for $1770, and pays $31 for each horse, and $21 for each ox. How many horses and oxen did he buy?

Ans. 9 and 71, or 30 and 40, or 51 and 9.

21) A person buys 124 head of cattle, viz. pigs, goats and sheep, for $400. Each pig cost $4½, each goat 3⅙, and each sheep $1¼. How many were there of each kind?

Ans. 17, 99, 8; or 40, 60, 24; or 63, 21, 40.

22) Divide 30 into three such parts, that if the first part be multiplied by 7, the 2d by 19, and the 3d by 38, the sum of the three products = 745. What parts are they?

Ans. 6, 11, 13.

23) Divide 100 into three such parts, that if the first be multiplied by 17, the 2d by 11, the 3d by 3, and the three products added together, the sum = 880. What parts are they?

Ans. 2, 69, 29; or 6, 62, 32; or 10, 55, 35, and so on; in all 10 different answers.

24) Required to find three whole numbers such, that if the first be multiplied by 5, the 2d by 13, the 3d by 18, the sum of the products = 997; but if the first be multiplied by 11, the 2d by 20, and the 3d by 37; then the sum of the products = 1866.

Ans. 16, 29, 30.

25) 30 persons, men, women, and children, spend, jointly, $58. Each man spent $3\frac{1}{2}$, each woman $1\frac{3}{8}$ dollars, and each child $\frac{1}{4}$ dollar. How many men, women, and children, were there?

Ans. 10 men, 16 women, and 4 children.

26) Required to find two numbers whose sum and product are equal?

Ans. If x and y, denote the two numbers sought, then x is arbitrary, and $y = \frac{x}{x-1}$.

27) Required to find two numbers whose sum is to their product as m to n.

Ans. If x and y denote the two numbers sought; then x is arbitrary, and $y = \frac{nx}{mx - n}$.

28) Required to find two whole numbers, whose sum and product are together = 139.

Ans. 1 and 69, 3 and 34, 4 and 27, 6 and 19, 9 and 13.

29) Required to find two whole numbers, whose product exceeds twice their difference by 100.

Ans. 10 and 10, 14 and 8, 22 and 6, 30 and 5, 46 and 4, 94 and 3.

30) How can we divide the fraction $\frac{230}{77}$ into two other fractions, whose denominators are 7 and 11?

Ans. Into $\frac{5}{7}$ and $\frac{25}{11}$, or in $\frac{12}{7}$ and $\frac{14}{11}$, or in $\frac{19}{7}$ and $\frac{3}{11}$.

31) Required to find two numbers, the sum of the squares of which is again a square.

Ans. If p and q denote two arbitrary numbers, then one of the numbers sought is $= p^2 - q^2$, and the other $= 2pq$, viz. 3 and 4, 6 and 8, 5 and 12, and so on.

32) Let a and c denote two rational numbers, what rational numbers can be assumed for x and y, so that the formula $a^2x^2 + cy^2$ may be a complete square?

Ans. $x = cn^2 - m^2$, $y = 2amn$; for we have

$$a^2(cn^2 - m^2)^2 + c(2amn)^2 = a^2(cn^2 + m^2)^2.$$

When a and c are given, arbitrary values may be assumed for m and n, by which the values of x and y may be determined.

33) What values may be assigned to the indeterminate quantity x, so that the formula $a^2x^2 + c$ may become a complete square?

Ans. $x = \dfrac{cn^2 - m^2}{2amn}$; because

$$a^2\left(\frac{cn^2 - m^2}{2amn}\right)^2 + c = \left(\frac{cn^2 + m^2}{2mn}\right)^2.$$

34) If a, b, c, denote three rational numbers, what ra-

tional numbers may be assumed for x and y, so that the formula $a^2x^2 + bxy + cy^2$ may become a complete square?

Ans. $x = m^2 - cn^2$, $y = bn^2 - 2amn$;
because we have

$$a^2(m^2 - cn^2)^2 + b(m^2 - cn^2)(bn^2 - 2amn) + c(bn^2 - 2amn)^2$$
$$= (am^2 - bmn + acn^2)^2.$$

35) What value can be assumed for x, so that the formula $b^2x^2 + bx + c$ may be a square?

Ans. $x = \dfrac{m^2 - cn^2}{bn^2 - 2amn}$; for

$$a^2\left(\frac{m^2 - cn^2}{bn^2 - 2amn}\right)^2 + b\left(\frac{m^2 - cn^2}{bn^2 - 2amn}\right) + c =$$
$$\left(\frac{am^2 - bmn + acn^2}{bn^2 - 2amn}\right)^2.$$

36) What value can be given to x, in order that the formula $ax^2 + bx + c^2$ may be a complete square?

Ans. $x = \dfrac{bn^2 - 2cmn}{m^2 - an^2}$; because

$$a\left(\frac{bn^2 - 2cmn}{m^2 - an^2}\right)^2 + b\left(\frac{bn^2 - 2cmn}{m^2 - an^2}\right) + c^2 =$$
$$\left(\frac{cm^2 - bmn + acn^2}{m^2 - an^2}\right)^2.$$

37) Let $x = w$ be such a value of x, that by substituting it for x, the formula $ax^2 + bx + c$ may become a complete square. How can more of such values be found?

Ans. If we substitute $w + py$ for x in the given formula, then it will become transformed into another of the form $fy^2 + gy + h^2$. This formula is a complete square when

$y = \frac{gn^2 - 2hmn}{m^2 - fn^2}$; consequently the given formula itself is a square when $x = w + \frac{p(gn^2 - 2hmn)}{m^2 - fn^2}$.

38) Let it be assumed that the formula $ax^2 + bxy + cy^2$ admits of being decomposed into two rational factors $mx + ny$, $m'x + n'y$; that, consequently, $b^2 - 4ac$ is a complete square. What numbers must be assumed for x and y, in order that the formula $ax^2 + bxy + cy^2$ may become a perfect square?

Ans. $x = np^2 - n'q^2$, $y = m'q^2 - mp^2$; because then

$$mx + ny = (m'n - mn')q^2 \text{ and}$$
$$m'x + n'y = (m'n - mn')p^2 \text{; consequently}$$
$$ax^2 + bxy + cy^2 = (mx + ny)(m'x + n'y)$$
$$= (m'n - mn')^2p^2q^2.$$

39) How is the equation $x^2 - \mathcal{A}y^2 = 1$ solved, when $\mathcal{A}$ denotes any other positive number than a square number; and it is required, that x and y should be whole numbers?

Ans. The solution of this important problem is already contained in No. 3 and 4, page 148. The values of x and y are no other than the numerator and denominator of the approximating values of the continued fraction there given. For example, when $\mathcal{A} = 106$, $x = 4005$, $y = 389$; when $\mathcal{A} = 124$, then $x = 4620799$, $y = 414960$; when $\mathcal{A} = 133$, then $x = 2588599$, $y = 224460$. These numbers, moreover, are the smallest numbers which can be found. (This method was first taught by *Lagrange;* it is by far the safest, and is much easier than *Pell's* method, given by *Euler*.

40) When $x = m$, $y = n$, are known values of x and y, and whole numbers, which solve the equation $x^2 - Ay^2 = 1$, in which A is a whole number; how can other integer values of x and y be found, which solve the same equation?

Ans. $$x = \frac{(m + n\sqrt{A})^p + (m - n\sqrt{a})^p}{2},$$

$$y = \frac{(m + n\sqrt{A})^p - (m - n\sqrt{a})^p}{2}.$$

The irrational part vanishes in the solution.

When $p = 0$, $x = 1$, $y = 0$

When $p = 1$, $x = m$, $y = n$, which are the values already known

When $p = 2$, $x = m^2 + An^2$, $y = 2mn$

When $p = 3$, $x = m^3 + 3Amn^2$, $y = 3m^2n + An^3$

When $p = 4$, $x = m^4 + 6Am^2n^2 + A^2n^4$,
$y = 4m^3n + 4Amn^3$,
and so on.

How can the problem, to make the formula

$$fx^2 + gxy + hy^2$$

a perfect square, (if this is at all possible) be identified with the problem to solve the equation $x^2 - Ay^2 = 1$ in whole numbers?

41) Required to find three numbers, such, that their sum, as well as the sum of every two of them, may be perfect squares?

Ans. 41, 80, 320; or 22, 42, $68\frac{1}{4}$, and an infinite number of others.

42) Required to find two numbers, such, that their difference equals the difference between their cubes?

Ans. $\frac{5}{7}$ and $\frac{3}{7}$, $\frac{8}{13}$ and $\frac{7}{13}$, $\frac{16}{19}$ and $\frac{5}{19}$, and an infinite number of others.

43) What remainders can a square number leave, when it is divided by 2, 3, 4, 5, 6, &c?

Ans. Divided by 2, 3, and 4, it leaves the remainders 0, 1; by 5, the remainders 0, 1, 4; by 6, the remainders 0, 1, 3, 4; by 7, the remainders 0, 1, 2, 4; by 8, the remainders 0, 1, 4; by 9, the remainders 0, 1, 4, 7; by 10, the remainders 0, 1, 4, 5, 6, 9; &c.

Can the formula $3x^n + 2$ ever be a perfect square, if whole numbers be assumed for x? Can the formula $14x^n + 3$ be a square?

44) What remainders will a cubic number leave, when divided by the numbers 7, 8, 9?

Ans. Divided by 7, the remainders 0, 1, 6; by 8, the remainders 0, 1, 3, 5, 7; by 9, the remainders 0, 1, 8.

Can the formulæ $8x^n + 6$, $18x^n + 7$, ever be perfect cubes, when whole numbers are assumed for x?

45) Find two numbers, such, that the sum of their squares may be a product of two factors, each of which is, again, a sum of two squares?

Ans. The analytic equation

$$(mm' + nn')^2 + (mn' - nm')^2 = (m^2 + n^2)(m'^2 + n'^2)$$

answers this question, in which, for m, n, m', n', all possible, whole, or fractional numbers may be assumed.

46) Find four squares, such, that their sum may be a product of two factors, one of which is a sum of three squares, and the other a sum of two squares?

Ans. The analytic equation

$$(mm' + nn')^2 + (mn' - nm')^2 + (pm')^2 + (pn')^2 = (m^2 + n^2 + p^2)(m'^2 + n'^2)$$

answers the conditions.

47) Find four squares, such, that their sum may be decomposed into two factors, each of which is a sum of three squares?

Ans. The analytic equation

$$(mm' + nn' + pp')^2 + (mn' - nm')^2 + (mp' - pm')^2 + (np' - pn')^2 = (m^2 + n^2 + p^2)(m'^2 + n'^2 + p'^2)$$

answers the conditions.

48) Find four squares, whose sum may be decomposed into two factors, each of which is, again, a sum of four squares?

Ans. The analytic equation

$$(mm' + nn' + pp' + qq')^2 + (mn' - nm' + pq' - qp')^2 + (mp' - pm' + qn' - nq')^2 + (np' - pn' + mq' - qm')^2 = (m^2 + n^2 + p^2 + q^2)(m'^2 + n'^2 + p'^2 + q'^2)$$

answers the question.

49) What values can we assign to the indeterminate quantities x, y, that the formula $x^2 + Ay^2$ may be a product of two factors of the *same form?*

Ans. $x = mm' + Ann'$, $y = mn' - nm'$; because

$$(mm' + Ann')^2 + A(mn' - nm')^2 = (m^2 + An^2)(m'^2 + An'^2).$$

50) Let a, a', a'', &c, be the remainders of the numbers A, A', A'', &c, divided by the same number k; then these numbers can be represented by the formulæ $nk + a$,

$n'k + a'$, $n''k + a''$, &c, and the remainder of the product $\mathfrak{A}\mathfrak{A}'\mathfrak{A}''$, &c, is consequently the same as that of the product $aa'a''$, &c. How, then, may, by this means, the remainder of a high power be ascertained, when divided by an arbitrary number?

Ans. Decompose the given power into lower ones, as factors, and seek the remainder from these; the product of these remainders divided by k, gives the remainder sought. Thus, you may proceed from the 2d power, and go on to the 4th, 8th, 16th, &c. For example, the remainder of 543^{113} divided by 257, or which is the same, the remainder of 29^{113}, is $= 57$.

* * *

"*Let p be a prime number, and $\mathfrak{A}$ another number, which is not divisible by p; then the power $\mathfrak{A}^{p-1}$ divided by p, will always leave the remainder* 1." How can this very important proposition in the theory of numbers, be demonstrated?

* * *

51) When p is a prime number, and m any very great number, how then, by means of this proposition, may the remainder of a power $\mathfrak{A}^m$ be obtained, still more briefly than in the preceding problem?

Ans. Let m divided by $p - 1$, give the remainder r; then the remainder of $\mathfrak{A}^m$ is the same as that of $\mathfrak{A}^r$, and $r < p - 1$.

52) In a number of the Berlin monthly journal I once found some treatises on the magnitude of the number which is represented by the double power $9^{9^9} = 9^{387420489}$. A great deal was said there on this subject; and this number

exceeds really, in magnitude, all conception; for it is, according to my calculation, expressible by no less than 369693100 digits, which may easily be found by logarithms. But what remainders will this vast number have, when it is divided by the prime numbers 11, 13, 17, 19?

Ans. By 11, the remainder 5; by 13, the remainder 3; by 17, the remainder 9; and by 19, the remainder 16.

53) What remainder does the power $A^{\frac{p-1}{2}}$ leave, when divided by p, when p is a prime number, and A is not divisible by p?

Ans. Either $+1$ or -1; consequently, either 1 or $p-1$.

54) When p is a prime number, and $a, b, c, \ldots\ldots t, u$, are whole, positive, or negative numbers, how many integer values of x are there, at most, between 0 and p, which can make the formula $ax^m + bx^{m-1} + cx^{m-2} + dx^{m-3} + \ldots tx + u$ divisible by p?

Ans. At most m, unless all the coefficients $a, b, \ldots u$, are, at the same time, divisible by p, in which case every value of x satisfies the condition. Why?

From these values of x, which render the above formula divisible by p, and which are limited between 0 and p, how can all other values, possessing the same property, be represented by a definite number of formulæ?

55) When for x, y, only whole rational numbers may be assumed, then the formula $ax^2 + bxy + cy^2$, in which a, b, c, are given numbers, cannot represent any whole number at pleasure, but only a certain class of whole numbers, which suit this formula. But how can such a formula be transformed into another $a'x'^2 + b'x'y' + c'y'^2$, which

possesses the property, that it represents all numbers which can be exhibited by the other, and inversely, that all numbers which this includes, may also be comprehended by the other?

Ans. Take any two whole numbers m, n, but which have no common divisor; determine afterwards two others m', n', which solve the indeterminate equation $mn' - nm' = \pm 1$, (of these we know there are an infinite number); then put $x = mx' + ny'$, $y = m'x' + n'y'$, and substitute these values in the formula $ax^2 + bxy + cy^2$; then this will be transformed into another $a'x'^2 + b'x'y' + c'y'^2$, which possesses the property required.

It is worthy of remark, that always $b^2 - 4ac = b'^2 - 4a'c'$. Why? The quantity $b^2 - 4ac$ is generally called, the *determining quantity*, because the nature of the formula depends upon it. It is also, at the same time, that quantity, by whose sign it may be perceived, whether the formula $ax^2 + bxy + cy^2$, may be decomposed into two real factors $kx + ly$, $k'x + l'y$, or not.

The Diophantic problems belong to that beautiful and interesting part of pure arithmetic, where the numbers are not considered with respect to their general relations, which they have as quantities, but with regard to the peculiar properties, by which they differ from one another. Whoever wishes to inform himself fully on this subject, may read the last chapter of Euler's Algebra, (translated into French by Garnier and Lagrange), the "Théorie des Nombres" of Legendre, and the "Disquisitiones Arithmeticæ" of Gauss.

In reading the two above-mentioned works of Gauss and Legendre, I consider one rule in particular, as worthy of remark; that which is known by the "Rule of Reciprocity," of which Gauss, besides the proof in his work, has also given another still more simple, which Legendre has adopted in the second edition of his work. Whether it is to be found in any other book besides, I am not aware. It is as follows:

If p and q be any two prime numbers, then the remainders arising from $q^{\frac{p-1}{2}}$ divided by p, is the same as that arising from the division of $p^{\frac{q-1}{2}}$ by q, viz. both $+ 1$ or both $- 1$, if the prime numbers p and q, are not both of the form $4n + 3$. If, on the contrary, they are both of the form $4n + 3$, then the remainders are the reverse of each other; viz. one is $+ 1$ and the other $- 1$, and vice versa.

The 55th problem is the basis of a very extensive theory of Trinomial Factors, on which subject there is, in the above two works, a great deal said which is partly new.

XX. PROBLEMS INVOLVING PROGRESSIONS AND FIGURATES.

1) A gentleman hires a servant, and promises him, for the first year, only \$30 in wages, but for each following year \$4½ more than for the preceding. How much will the servant receive for the 17th year of his engagement? and how much for all 17 years together?

Ans. \$102, and \$1122.

2) If \$3500 be lent out at 4 per cent interest, and each year \$300 be added to it: how much will the interest amount to at the end of 24 years?

Ans. \$6672.

3) A traveller who wishes to be at the place of his destination in 19 days, expedites his journey so much, that he goes each day a quarter of a mile more than he did the preceding. If, then, he travels $14\frac{1}{2}$ miles the last day, how many miles did he go the first day? And how many miles did his whole journey amount to?

Ans. 10 miles, and $232\frac{3}{4}$ miles.

4) What is the difference of an arithmetical progression of 22 terms, whose first term is 1, and last term 15?

Ans. $\frac{2}{3}$.

5) How many terms does an arithmetical progression consist of, whose difference is 3, first term 5, and last term 302?

Ans. Of 100 terms.

6) A person being asked about his salary, answered, "At present I have $550; but when I first entered into office I had no more than $100; but, on account of my industry, I received, each year, an addition of $30 to my income." How long was he engaged?

Ans. 16 years.

7) A debtor being unable to pay his debt, amounting to $12990, at once, agrees with his creditors to discharge it by monthly instalments, viz. $600 the first month, but each succeeding month $50 more than the preceding one. In how many months will he have discharged his whole debt? And how much does he pay the last month?

Ans. In 14 months, and $1250.

8) We know, from natural philosophy, that each body, which falls *in vacuo,* passes, in the first second of its fall, through a space of $16\frac{1}{12}$ feet; but in each succeeding second,

32⅙ feet more than in the immediately preceding one. Now, if a body has been falling 20 seconds, how many feet will it have fallen the last second? And how many in the whole time?

Ans. 627¼ feet, and 6600 feet.

9) How long, according to the conditions of the foregoing problem, will a body fall, to accomplish a space of 4116 feet?

Ans. 16 seconds.

10) In a company, the conversation turning on domestic œconomy, a person said, "This year I have saved $78; I have also, already made a fortune of $1350 by the following means; since I first entered into employment till now, I have saved each year, of my salary, $2 more than in the preceding." How long was the person in office? And how much did he save the first year?

Ans. 25 years, and $30.

11) A person was fined $800 by the judge, which were to be paid by instalments; the first one was $20, but in each succeeding period something invariably greater than in the preceding, so that in the last period he paid $80. In how many payments will the whole fine be discharged? And what was the common difference between each successive payment?

Ans. 16 instalments, and $4.

12) In a foundery, a person saw 15 rows of cannon-balls placed one above another, and asked a bombardier who stood near him, how many balls there were in the lowest row. "You may easily calculate that," answered the bombardier, "in all these rows together, there are 4200 balls, and each row, from the first to the last, contains 20 balls less than the

one immediately below it." How many balls, therefore, were there in the lowest row?

Ans. 420.

13) A meteorologist observed this remarkable phenomenon, that from the 8th to the 19th of June, of a certain year, the thermometer rose half a degree daily, and that the arithmetic mean of these 12 different positions of the thermometer was 18¾ degrees. At what degree did it stand the 8th day?

Ans. At the 16th degree.

14) A company of soldiers who had successfully stormed a fortress, was rewarded as follows: That soldier who had mounted the wall first, received a certain sum of money; the 2d a little less; the 3d exactly so much less than the 2d, and so on. When the money was divided, two of the soldiers could not be present, on account of their wounds; their share was consequently given to two of their comrades; these two put their own money, and that of their two comrades, into the same purse; and afterwards, when they came to divide it, they had forgotten what fell to each man's share. One of them had received $92 for himself and his comrades, and only remembered, that he himself was the 2d, and his comrade the 7th. The other had received $71 for himself and his comrade, and knew that he himself was the 11th, and his comrade the 4th. How much did each of the four soldiers receive?

Ans. The 2d $54¾, the 4th $47¾, the 7th $37¼, and the 11th $23¼.

15) A person has 18 numbers, which are in arithmetical

progression. If the two middle ones be added together, their sum is $31\frac{1}{2}$; but if the first and last are multiplied together, their product is $85\frac{1}{2}$. What is the first number? and what the common difference?

Ans. 3, and $1\frac{1}{2}$.

16) Two persons wishing to have an interview, travel at the same time, on the same road, from two different places, A and B, which are 170 miles distant from one another. He who starts from B, travels regularly 4 miles a day; but the other only 2 miles the first day, and each following day he goes, without interruption, $\frac{1}{2}$ mile more than the preceding one. Where will they meet?

Ans. 102 miles from A.

17) A person enjoys a yearly income of $500, none of which he spends, but from the day he receives it annually, beginning from the day when he drew it for the first time, he immediately put it out at 5 per cent, and leaves this interest also untouched. In how many years will he thus accumulate $6875, at simple interest.

Ans. In 10 years.

18) In arsenals the cannon-balls are sometimes piled up in triangular pyramids, viz. in such a way, that the uppermost ball rests on the top of 3 balls, these 3 on 6, these 6 balls again on 10; and so on; or, in short, so that the numbers of balls in the different layers, calculated from the first, are represented by the triangular numbers 1, 3, 6, 10, 15, 21, 28, &c. How many balls, then, are there in the lowest layer of the triangle, if the side contain n balls? And how many balls are there in the whole pyramid?

Ans. $\frac{n(n+1)}{1\,.\,2}$ in the lowest layer, and

$\frac{n(n+1)(n+2)}{1\,.\,2\,.\,3}$ in the whole pyramid.

Example. When $n = 30$, the lowest layer contains $\frac{30,\ 31}{1\,.\,2} = 465$ balls, and the whole pyramid $\frac{30,\ 31,\ 32}{1\,.\,2\,.\,3}$ $= 4960$ balls.

19) But when the pyramid, in the preceding problem, is incomplete, and each side of the highest layer contains m balls. How many balls are there, in this case, in the pile?

Ans. $\frac{n(n+1)(n+2)}{1\,.\,2\,.\,3} - \frac{m(m+1)(m+2)}{1\,.\,2\,.\,3}$.

20) Sometimes, however, the balls are piled in quadrilateral pyramids, so that all the layers form squares, and the balls in the different layers, beginning from the top, are represented by the square numbers 1, 4, 9, 16, 25, 36, &c. If, then, there be n balls in each side of the lowest layer; consequently, n^2 balls at the bottom, how many are there in the complete pyramid? and how many in the incomplete pyramid, when the side of the highest layer contains m balls?

Ans. $\frac{n(n+1)(2n+1)}{1\,.\,2\,.\,3}$ in the complete pyramid; and

$\frac{n(n+1)(2n+1)}{1\,.\,2\,.\,3} - \frac{m(m+1)(2m+1)}{1\,.\,2\,.\,3}$

in the incomplete one.

21) When there are many balls to pile up, it is customary

to make the piles rectangular. In the upper row there are, then, only m balls; these rest on two layers, each of which contains $m + 2$ balls, and so on; viz. in each succeeding layer a row more, and in each row, one ball more. By this arrangement, the layers contain successively the following number of balls: m, $2(m + 1)$, $3(m + 2)$, and so on; consequently, there are in the nth layer $n(m + n - 1)$. How many balls, then, are there in the whole pile of n rows?

Ans. $\frac{n(n + 1)(2n + 3m - 2)}{1 \,.\, 2 \,.\, 3}$.

22) From the rectangular rows it is usual to form also another kind of piles, which, however, to retain their equilibrium, require, that on two sides, they should lean on two other piles, or be supported in any other way. Thus, if there be, for instance, m balls in the top row, there are under this 2 rows, each containing $(m - 1)$ balls; under these, 3 rows, each containing $m - 2$ balls, and so on; so that the balls in the different rows are represented by the following numbers: m, $2(m - 1)$, $3(m - 2)$, and so on; and consequently, in the nth layer, there are $n(m - n + 1)$ balls. How many balls are there in such a pile, consisting of n layers?

Ans. $\frac{n(n + 1)(3m - 2n + 2)}{1 \,.\, 2 \,.\, 3}$.

23) If a pile of this sort is only supported on one side, then each succeeding layer contains one row more, but the number of balls in each row remains the same. Then the number of balls in the different layers will be successively represented by the following numbers: m, $2m$, $3m$, $4m$, &c;

consequently, in the nth layer, there are nm balls. How many balls are there in a pile of this kind?

Ans. $\frac{mn(n+1)}{1\,.\,2}$.

24) When the number of balls in a complete triangular pyramid $= s$, what equation shall we have to solve, in order to determine from it the number of layers, or the number of balls, in the side of the lowest layer?

Ans. The equation $n^3 + 3n^2 + 2n - 6s = 0$.

25) But what equation for a four-sided pyramid?

Ans. $2n^3 + 3n^2 + n - 6s = 0$.

26) A king in India, named Sheran, wished, according to the Arabic author Asephad, that Sessa, the inventor of chess, should himself choose a reward. He requested the grains of wheat, which arise, when 1 is calculated for the first square of the board, 2 for the 2d, 4 for the 3d, and so on; reckoning for each of the 64 squares of the board, twice as many grains as for the preceding. When it was calculated, to the astonishment of the king, it was found to be an enormous number. What was it?

Ans. 18446744073709551615; a sum, which, according to a moderate calculation, could not be obtained upon the whole earth, in 70 years and upwards, if all the land were employed in the cultivation of wheat.

27) A person sowed a peck of wheat, and used the whole crop for seed the following year; the produce of this 2d year again for seed the 3d year, and so on. If, in the 10th year, his harvest is 1048576 pecks, by how many times must the

seed have increased each harvest, supposing the increase to have been always the same?

Ans. 4 times.

28) In a country, during peace, the population increased every year in the same proportion, and so fast, that in the space of 4 years, it became from 10000 to 14641 souls. By what part did the population increase yearly?

Ans. By the 10th part.

29) It is required to insert 10 terms between 1 and 3, so that there may result a geometrical progression of 12 terms. What is the 2d term of this progression?

Ans. $3^{\frac{1}{11}} = \sqrt[11]{3} = 1{,}105\ldots.$

30) What exponent has a geometrical progression of 23 terms, whose first term is 5, and last term 80? What is the sum of this progression? and what its 20th term?

Ans. The exponent is 1,0935.....; the sum 881,62.....; the 20th term 27,351.....

31) There are seven numbers, which form a geometrical progression, and which are such, that if the first 6 are added together, the sum is $157\frac{1}{2}$; but when the last 6 are added together, the sum is 315. What are the numbers?

Ans. $2\frac{1}{2}$, 5, 10, 20, 40, 80, 160.

32) In a geometrical progression of five terms, we know that the sum of the even terms $= a$, and the sum of the odd terms $= b$. What is the progression?

Ans. Let A, B, C, D, E, be the five terms of the sought progression; then the middle term

$$C = \frac{-b \pm \sqrt{(b^2 + 4a^2)}}{2};$$

$$\text{whence } A = \frac{[a-\sqrt{(a^2-4C^2)}]^2}{4C},$$

$$B = \frac{a-\sqrt{(a^2-4C^2)}}{2},$$

$$D = \frac{a+\sqrt{(a^2-4C^2)}}{2},$$

$$E = \frac{[a+\sqrt{(a^2-4C^2)}]^2}{4C}.$$

33) The sum of the even terms of a geometrical progression is $= a$, the sum of the odd terms $= b$. What is this progression?

Ans. The exponent of the progression is $= \frac{a}{b}$, the first term $= \frac{b3}{a^2+b^2}$; whence the rest is easily found.

34) The sum of the even terms of a geometrical progression of $2n$ terms is $= a$, the sum of the odd terms $= b$. What is the progression?

Ans. The exponent is $= \frac{a}{b}$; the first term is

$$= \frac{b^{2n-1}}{a^{2n-2} + a^{2n-4}b^2 + a^{2n-6}b^4 + \ldots\ldots b^{2n-2}}$$

$$= \frac{b^{2n-1}(a^2-b^2)}{a^{2n}-b^{2n}}.$$

35) In a geometrical progression of six terms, we know the sum of the two middle terms $= a$, the sum of the 1st and 6th $= b$. What is the progression?

Ans. Let the product of the two middle terms be $= p$;

then $p = \frac{a^2}{2} \cdot \frac{5a \pm \sqrt{(4ab + 5a^2)}}{5a - b}$. From p and a, the two middle terms may be obtained, which are

$$\tfrac{1}{2}[a - \sqrt{(a^2 - 4p)}], \quad \tfrac{1}{2}[a + \sqrt{(a^2 - 4p)}];$$

and finally from these the progression itself.

36) In a geometrical progression, we know that the sum of all the terms $= a$, the sum of their squares $= b$, and the sum of their cubes $= c$. How is it formed?

Ans. The exponent e of the progression is

$$= \frac{a^4 + 2ac - 3b^2 \pm \sqrt{12a(ac - b^2)(a^2 - c)}}{a^4 + 3b^2 - 4ac};$$

the first term is $= \frac{b(1 + e) + a^2(1 - e)}{2a}$. Now, from the exponent and the first term, and the sum of all the terms, the number of terms, and thereby the progression itself, may easily be found.

37) In a geometrical progression, the sum of all the terms $= a$, the sum of their squares $= b$, and the sum of their 4th powers $= c$. How is this progression found?

Ans. The exponent e of the progression is

$$= \frac{a^4 b - b^3 \pm 2a\sqrt{(c - a^2 b)(b^3 - ac^2)}}{b^3 + a^4 b - 2a^2 c};$$

the first term is $= \frac{b(1 + e) + a^2(1 - e)}{2a}$; whence the number of terms, and the progression itself, may be found.

38) The common difference of an arithmetical progression is $= d$, the product of all terms $= a$. How is the first term found?

Ans. Let the first term $= x$, the product of the two extreme terms $= y$; then $x^2 + 3dx = y$, and the unknown quantity y itself, is determined by the equation $y^2 + 2d^2y = a$. From these two equations we find the following four values of x:

$$x = -\tfrac{3}{2}d \pm \sqrt{[\tfrac{5}{4}d^2 + \sqrt{(a + d^1)}]}$$
$$x = -\tfrac{3}{2}d \pm \sqrt{[\tfrac{5}{4}d^2 - \sqrt{(a + d^4)}]}.$$

39) Let the difference of an arithmetical progression of six terms $= d$; the product of all the terms $= a$. How is the first term found?

Ans. Let the first term $= x$; the product of the two extremes $= y$; then $x^2 + 5dx = y$, and the unknown quantity y, is found from the equation

$$y^3 + 10d^2y^2 + 24d^4y = a.$$

In general, the problem "To find an arithmetical progression of $2m$ terms with the given difference d, so that the product of all its terms may be equal to a given quantity a," may be reduced to the solution of an equation of the 2d and mth degree. Then, it is only necessary to put

$$x^2 + (2m - 1)dx = y;$$

and we shall find for y an equation of the mth degree.

XX. PROBLEMS RELATIVE TO INTEREST AND ANNUITIES.

1) A capital a is at interest; the rate of interest $= p$. What will this capital amount to in n years?

Ans. ap^n.

2) A capital of $5000 stands at 4 per cent compound interest. What will it amount to in 40 years?

Ans. $24005,103.

3) What will $3200, at 3 per cent compound interest, amount to in 80 years?

Ans. $34050,84....

4) In a certain province there are at present 200000 inhabitants. If the population increases $\frac{1}{30}$ part yearly, what will it be in 100 years hence?

Ans. 14489276 nearly.

5) What will a capital of $12000 amount to in 10 years, if it bear 6 per cent, and the interest be paid half-yearly?

Ans. $21673,333....

6) How long must $3600 remain at 5 per cent, in order to accrue as much as $5000 at 4 per cent in 12 years?

Ans. Nearly 16 years?

7) How long must a capital a, remain at the interest p, to become as large as a capital a', at the interest p' in n years?

$$Ans.\ \frac{\log a' + n \log p' - \log a}{\log p} \text{ years.}$$

8) What must be the amount of a capital at 4 per cent,

that 15 years hence, it may be equal in value to \$4500 at 6 per cent for 9 years?

Ans. About \$4221.

9) What is the amount of a capital, which at the interest p for n years, is of equal value with a capital a' for n' years, at the interest p'?

Ans. $\log a = \log a' + n' \log p' - n \log p$. By means of this equation, first the $\log a$, and thence the capital a, may be found.

10) On the other hand, what is the rate of interest p, that a capital a, in n years, may be equal to a capital a' in n' years, at the interest p'.

$$Ans.\ \log p = \frac{\log a' + n' \log p' - \log a}{n}.$$

11) A debt of \$7963 is paid interest for at 5 per cent. If, 5 years hence, \$576 be paid off, and after 8 years \$498 more, what will be the amount of the remaining debt 10 years hence, if compound interest be reckoned?

Ans. Nearly \$11696.

12) What does the half-year's interest of a capital amount to, if 5 per cent, yearly, be paid, reckoning compound interest? Farther, what is the amount of a quarter's interest, on the same supposition?

Ans. The half year's interest is 2,4695, and the quarter's 1,2272 per cent, nearly.

13) What is the amount of $\frac{1}{n}$ part of a year's interest, at the rate p?

Ans. $100(\sqrt[n]{p} - 1)$ per cent.

14) How long must a capital stand at 4 per cent compound interest, that it may double itself? And how long that it may triple itself?

Ans. It doubles itself, in between 17 and 18 years, and triples itself, in between 28 and 29 years.

15) How long must it stand at p per cent, in order that it may be m times as great?

Ans. $\frac{\log m}{\log p}$ years.

16) A usurer lent a person $600, and drew up for the amount, a bond of $800, payable in 3 years, bearing no interest. What did he take per cent, if compound interest be taken into consideration?

Ans. Somewhat above 10 per cent.

17) If the capital a, in n years, be increased to A dollars, what is the rate of interest?

Ans. $\log p = \frac{\log A - \log a}{n}$.

(where p denotes the rate of interest).

18) A person has to pay a sum of $3750, six years hence. How much must he pay, in cash, for this sum, reckoning 4 per cent discount, and also compound interest?

Ans. About $2963,666.....

19) What is the present value of a capital a, which is due n years hence, computed at p per cent, compound interest?

Ans. $\frac{a}{p^n}$.

20) What is the present value of a stock of turf, which, 20 years hence, produces a sum of $500, so that the money laid down for it may bear 4 per cent?

Ans. $228.

21) A town contains 20000 souls, and we know that the population has regularly increased $\frac{3}{100}$ yearly. What was its population 10 years ago?

Ans. 14882.

22) Three persons bid at an auction. The first offered $30000 in cash; the second offered $33500 in three years, without interest; the third offered to pay $40000 in seven years, also without interest. Which of these three offers is the greatest, calculating 5 per cent compound interest? And by how much does it exceed the two others in value?

Ans. The first is the greatest, and exceeds the 2d by $1061, and the 3d nearly by $1573.

23) In how many years will the population of a place become 10 times as great as it is at present, if the yearly increase amount to three persons in a hundred?

Ans. In about 78 years.

24) A capital of $800, which was well laid out, increased in the space of 6 years, to $3600. What did the capital gain per cent.

Ans. $28\frac{1}{2}$, nearly.

25) A capital *a* is put out at the interest *p*, and at the expiration of each year, the interest is added to the principal; and at the same time it increases or diminishes yearly by the same sum *b*. What will this capital amount to in *n* years hence?

Ans. It is $= ap^n \pm \frac{b(p^n - 1)}{p - 1}$; where the upper sign $+$ belongs to the increased, and lower $-$ to the diminished capital.

26) A capital of $6000 stands at 5 per cent compound interest, and each year, besides the interest added to the capital, it is increased by $500. How much will the capital, by these means, become in 10 years?

Ans. About $16062,291....

27) What will a capital of $3740 amount to in 8 years, if at the end of each year $450 be added to it, reckoning 4 per cent compound interest?

Ans. About $9264,833.

28) A debt of $15467 is paid interest for at the rate of 5 per cent. At the end of each year $600 are deducted. What will be the amount of the remainder of the debt, at the expiration of 10 years?

Ans. 17647, nearly.

29) From a capital of $5000, standing at 5 per cent interest, a person takes away $400 yearly. What will the remainder be 10 years hence?

Ans. About $3113.

30) A person is bound, for 7 years successively, to pay, at the beginning of each year, $4000, but is behindhand with the payment. What will he owe at the beginning of the 7th year, if the compound interest be calculated at 4 per cent.

Ans. $31593, nearly.

31) A person puts a capital of $30000 out at 4 per cent; takes, at the end of each year, from the interest he received,

$800, for his support, and adds the remainder to the capital. What will the amount of his capital be 15 years hence?

Ans. $38009,416....

32) A person has put his whole fortune of $100000 out at 5 per cent interest; but is unable to pay his expenses with the interest of this capital, because he requires for this purpose $6000 yearly. He is, consequently, at the end of each year, obliged to withdraw as much from his capital, as will, with the interest he receives, make up the $6000. If he continues thus, in how many years will he be a beggar?

Ans. In from 36 to 37 years.

33) A capital a is lent out at the rate of p per cent. In what time will it become a', if the capital, with the interest and compound interest added to it, be augmented or diminished yearly by the sum b?

$$\textit{Ans.}\quad n = \frac{\log[(p-1)a' \pm b] - \log[(p-1)a \pm b']}{\log p}$$

(n stands for the number of years). The upper sign $+$, denotes the addition of b, the lower $-$, the deduction of b.

34) If, in the preceding problem, b is yearly taken away, and b is greater than the interest of the capital a; in this case, in how many years will the capital be spent?

$$\textit{Ans.}\quad n = \frac{\log b - \log[b - (p-1)a]}{\log p},$$

gives the number of years.

35) What is the present value w, of an annuity r, at p per cent, which a person has to enjoy n years?

$$\textit{Ans.}\quad w = \frac{(p^n - 1)r}{(p-1)p^n}.$$

36) A person who has to enjoy an annuity of $500 for 6 years, wishes to sell it. How much cash can a person give him for this annuity, calculating the compound interest at $3\frac{1}{2}$ per cent?

Ans. 2664,291....

37) What is the present value of an annuity of $350, assigned for 8 years, calculating 4 per cent interest?

Ans. A little over $$2356\frac{1}{24}$.

38) How much can a person give for an annuity of $400, which has to run 12 years, if the interest be reckoned at 3 per cent?

Ans. A little over $$3981\frac{7}{12}$.

39) If an annuity *r*, having *n* years to run, be worth in cash *w*, what is its amount?

$$Ans.\ r = \frac{(p-1)p^n w}{p^n - 1}.$$

40) A debt, due at this present time, amounting to $1200, is to be discharged in seven yearly and equal payments. What is the amount of one of these payments, if the interest be calculated at 4 per cent?

Ans. Nearly $200.

41) What is the amount of that annuity, the receipt of which, for 13 years, is considered as equal in value to $20000 cash, computing the interest at 4 per cent?

Ans. $2003, nearly.

42) A company borrowed of the government $20000, and for this sum mortgaged a forest, which yielded a clear annual income of $1500. How long can the government

retain this forest for the capital advanced, if the interest be calculated at 5 per cent?

Ans. 22 years nearly.

43) A person wishes to obtain an annuity of $2000 for $34580. For how many years can this annuity be granted him, computing the interest at 4 per cent?

Ans. For about 30 years.

44) How long has an annuity to run, if its present value be considered as equal to w dollars, the interest being 5 per cent?

$$Ans.\ \log n = \frac{\log r - \log\left[r - (p-1)w\right]}{\log p}.$$

45) An annuity r' is required for the given space of n' years, whose *present* value is equal to another annuity r for n years. If both be calculated at p per cent, what is the annuity r'?

$$Ans.\ r' = \frac{(p^n - 1)p^{n'-n}}{p^{n'} - 1}\,r.$$

46) But how great is the required annuity, if it be payable in m years hence; that is, if it is to be paid for the first time m years hence, and then to be continued during n' years?

$$Ans.\ r' = \frac{(p^n - 1)p^{n'-n+m}}{p^{n'} - 1}\,r.$$

47) What is the present value of an annuity r, which rises in a geometrical progression, whose exponent is e, if it continues for n years, at p per cent?

$$Ans.\ \frac{r(e^n - p^n)}{(e-p)p^n},\quad \text{or}\quad \frac{r\left(\dfrac{e^n}{p^n} - 1\right)}{e-p}.$$

The latter formula is more easily calculated by logarithms.

48) What is the present value of an annuity for n years, if the increase be in the arithmetical progression r, $2r$, $3r$, $4r$, &c, so that at the end of the first year r be paid, at the end of the 2d, $2r$, and so on, calculating p per cent?

Ans. Let the present value of an annuity r, which always remains the same, for n years, be $= w$, then the present value of an annuity, increasing in the given arithmetical progression, is

$$= \frac{1}{p-1}\left(pw - \frac{nr}{p^n}\right).$$

The following observation is intended for the teacher. The last problem may be very easily solved by the help of geometrical progressions. It is, however, only a particular case of a much more general one. Let $A + Bx + Cx^2 + \ldots$ $\ldots$ &c, be the increased annuity after the undetermined xth year; then the present value of this single increase $= p^{-x}(A + Bx + Cx^2 \ldots\ldots\ldots\ldots$ &c); and, consequently, $Sp^{-x}(A + Bx + Cx^2 + \ldots\ldots\ldots$ &c), is the present value of the annuity, where S is the symbol of summation; and summations of this kind are always practicable. (See Traité du calcul des différences et des séries, par Lacroix, p. 90, § 911, the first edition). In the above problem $B = r$, and A, C, D, &c, $= 0$.

XXII. PROBLEMS RELATING TO PERMUTATIONS, COMBINATIONS, AND VARIATIONS.

1) How often can eight persons, standing near one another, change their places, that is, so change them, that they may present another order each time?

Ans. 40320 times.

2) How often can the 24 letters of the alphabet be transposed?

Ans. 620448401733239439360000 times. All the inhabitants of the globe, on an average, could not, in a thousand millions of years, write out all the transpositions of the 24 letters, even supposing that each wrote 40 pages daily, each of which pages containing 40 different transpositions of the letters.

3) How many changes are there contained in all the permutations of *abcdefg*, which begin with one of the letters *a*, *b*, *c*, *d*, *e*, *f*, *g*?

Ans. 720.

4) How many do they contain beginning with *ab*? How many with *abc*? How many with *abcd*?

Ans. 120 with *ab*, 24 with *abc*, and 6 with *abcd*.

5) How many do they contain, in which the letters *abcd*, remain together in the same order, in which they stand here?

Ans. 24.

6) On the other hand, how many do they contain, in which the letters a, b, c, d, are placed without any regard to their order?

Ans. 576.

7) How many changes are there contained in all the transpositions of $a^3 b^5 c^4$ which begin with c^3?

Ans. 504.

8) How many, which begin with $a^2 b^2 c$?

Ans. 140.

9) How many are there among them, in which a occupies a fixed place, say the fourth?

Ans. 6930.

10) What is the sum of all the figures, in all the permutations of 12234, taken together?

Ans. 720.

11) How many combinations of two, three, four, and five numbers, are there contained in 90 numbers?

Ans. 4005 combinations of two and two, 117480 of three, 2555190 of four, and 43949268 of five numbers.

12) How many combinations of two, three, four, and five numbers, are there contained in 60 numbers?

Ans. 1770 combinations of two, 34220 of three, 487635 of four, and 5461512 of five numbers.

13) From a pack of piquet cards, consisting of 32 cards, well shuffled, a person blindfold, and without picking, draws 15 cards. In how many different ways may the cards so drawn, be arranged?

Ans. In 565722720 different ways.

14) The product *abcd* may be decomposed into 3 different ways, into smaller products, each consisting of two factors, viz. into $ab \times cd$, $ac \times bd$, $ad \times bc$, the product *abcdef*, admits of 15 such divisions, viz. $ab \times cd \times ef$, $ab \times ce \times df$, $ab \times cf \times de$, &c. In how many different ways can the product *abcdefg* ..... &c, containing $2n$ factors, be divided into such products of two factors?

Ans. In $\frac{(n+1)(n+2)\ldots\ldots(2n-1)2n}{(1\,.\,2)^n}$ ways.

15) In how many different ways can the product *abcdef* ..... &c, composed of $3n$ factors, be decomposed into smaller products of 3 factors each?

Ans. In $\frac{(n+1)(n+2)\ldots\ldots(3n-1)3n}{(1\,.\,2\,.\,3)^n}$ ways.

16) In how many different ways can the product *abcde* ..... &c, composed of mn factors, be decomposed into smaller products, each containing m factors?

Ans. In $\frac{(n+1)(n+2)\ldots\ldots(mn-1)mn}{(1\,.\,2\,.\,3\ldots\ldots m)^n}$ ways.

17) Every whole number N, is either itself a prime number, or is a product of equal or different prime numbers. Thus, if $a, b, c, \ldots\ldots$ &c, denote prime numbers, then, in general, $N = a^m b^n c^p$, &c, where m, n, p, &c, can represent all possible whole numbers, 0 included. Now, how can you find all possible numbers, by which the number N is divisible? And how many divisors will it, generally, have?

Ans. For shortness' sake, let $1 + a + a^2 + \ldots\ldots a^m = A$, $1 + b + b^2 + \ldots\ldots b^n = B$, $1 + c + c^2 + \ldots\ldots c^p$, &c, $= C$, then the terms of the product $ABCD$ ..... give all the divisors of the number N, and their number itself, is

$$=(m+1)(n+1)(p+1)\ldots\ldots\&c.$$

For instance, 360 being $=2^3.3^2.5^1$, the product of $4.3.2$ gives the number of divisors of 360, and the terms of the product $(1+2+4+8)(1+3+9)(1+5)$ give the divisors themselves, viz. 1, 2, 4, 8, 3, 6, 12, 24, 9, 18, 36, 72, 5, 10, 20, 40, 15, 30, 60, 120, 45, 90, 180, 360.

18) How may the numbers be found, which have exactly n divisors, neither more nor less?

Ans. Let a, b, c, &c, be any prime numbers. Now, if n is a prime number, a^{n-1} is such a number as is required. If n is a compound number, decompose it into the factors, m', m'', m''', &c, (it matters not whether m', m'', m''', are themselves prime numbers or not); then $a^{m'-1}\,b^{m''-1}\,c^{m'''-1}$, &c, will be such a number as required.

19) A complete polynomial of the Nth dimension for n quantities x, y, z, &c, is that which contains all combinations of n quantities, from the 0th to the Nth class, inclusively. Thus, for instance, a polynomial of the 4th dimension for two quantities is this: $a+bx+b'y+cx^2+c'xy+c''y^2+dx^3+d'x^2y+d''xy^2+d'''y^3+ex^4+e'x^3y+e''x^2y^2+e'''xy^3+e''''y^4$. How many terms does a complete polynomial of the Nth dimension for n quantities consist of?

Ans. Of $\dfrac{(N+1)(N+2)(N+3)\ldots\ldots(N+n)}{1\;.\;2\;.\;3\ldots\ldots\ldots\ldots n}$ terms.

20) How many of these terms remain, when all those which are divisible by x^p are excluded?

$$Ans.\ \frac{(N+1)\,(N+2)\,(N+3)\,\ldots\ldots\,(N+n)}{1\ .\ 2\ .\ 3\ \ldots\ldots\ldots\ldots\ n}$$

$$-\frac{(N-p+1)\,(N-p+2)\,(N-p+3)\ldots\ldots(N-p+n)}{1\ .\ 2\ .\ 3\ \ldots\ldots\ldots\ldots\ n}$$

21) But how many remain, when all those which are divisible by x^p and y^q are excluded?

$$Ans.\ \frac{(N+1)\,(N+2)\,(N+3)\,\ldots\ldots\,(N+n)}{1\ .\ 2\ .\ 3\ \ldots\ldots\ldots\ldots\ n}$$

$$-\frac{(N-p+1)\,(N-p+2)\,(N-p+3)\ldots\ldots(N-p+n)}{1\ .\ 2\ .\ 3\ \ldots\ldots\ldots\ldots\ n}$$

$$-\frac{(N-q+1)\,(N-q+2)\,(N-q+3)\ldots\ldots(N-q+n)}{1\ .\ 2\ .\ 3\ \ldots\ldots\ldots\ldots\ n}$$

$$+\frac{(N-p-q+1)\,(N-p-q+2)\ldots\ldots(N-p-q+n)}{1\ .\ 2\ \ldots\ldots\ldots\ldots\ldots\ldots\ n}$$

22) Let it be assumed, that in n several events which may happen, there are m cases, which are favourable to any hope or expectation founded upon them; consequently, $n - m$ cases, where an unfavourable result may be expected. What probability is there for a favourable, and what for an unfavourable event?

Ans. $\frac{m}{n}$ for the favourable, and $\frac{n-m}{n}$ for the unfavourable event.

What do these expressions mean when $m = 0$, or when $m = n$?

23) In a wheel of fortune, containing 13 different numbers, what proportion does the probability of drawing 6 numbers, previously agreed upon, bear to the probability of drawing from a pack of piquet cards, (32 in number) 8 cards,

and 10 red; at first, there are 7 balls drawn blindfold, and then again, from the remaining 17, three more. I bet that all the 7 balls are red, and the three last are all black. What is my chance of winning?

Ans. $\frac{5}{490314}$, and consequently but little.

34) How many different throws can be made with 2, 3, 4 ..... and, in general, with *n* dice?

Ans. With two dice 36, with three 216, with four 1296, and, in general, with *n* dice, 6^n throws.

35) What is the probability of throwing with four dice, the same number on all four?

Ans. $\frac{1}{216}$.

36) What is the probability, with three dice, of throwing two alike, and no more?

Ans. $\frac{5}{12}$.

37) What is the probability, with four dice, of throwing two numbers alike, so that the two remaining dice may have unlike numbers? and what is the probability, on the same condition, for 5, 6, and 7 dice? Lastly, what is it for 8 dice?

Ans. For four dice the probability sought is $= \frac{5}{9}$; for five dice it is $= \frac{25}{54}$, for six $= \frac{28}{108}$, for seven $= \frac{35}{648}$, and for eight dice it is $= 0$; that is, the case is impossible.

38) What is the probability, with four dice, of throwing a triplet, or of throwing three number of the same kind, and no more?

Ans. $\frac{5}{54}$.

39) What is the probability, with five dice, of throwing three numbers alike, and no more; but in such a way, that

the other dice have different numbers? What is it, on the same condition, for 6, 7, and 8 dice? Lastly, what is it for 9 dice?

Ans. For five dice the probability is $= \frac{25}{162}$, for six dice also $= \frac{25}{162}$, for seven dice $\frac{175}{1944}$, for eight dice $= \frac{35}{1458}$, and lastly, for nine dice it is $= 0$, that is, the case is impossible.

40) Which is greater, the probability of throwing two sixes with three dice, or from a shuffled pack of piquet cards, to draw three cards of the same colour?

Ans. The first is the more probable, and the probability of the first case is to that of the 2d as 755 to 504.

41) There are two games offered me to bet upon. In the one, eight equal numbers, and no more, were to be thrown by 12 dice, but at the same time, in such a way, that the remaining four are all different numbers. In the other, I am to draw from a whole pack of cards (52 in number) five cards of the same colour. What game, if I were disposed to bet, ought I to give the preference to? And what proportion do the chances of winning at these games bear to one another?

Ans. The second chance is better than the first, and the probability of winning at it, is to that of winning at the first, as 1259712 is to 104125, or about as $12\frac{1}{10}$ is to 1.

XXIII. MISCELLANEOUS PROBLEMS.

1) From a pack, consisting of 32 cards, three cards were drawn; upon each of these there were as many cards laid as were requisite that their number, together with the number of pips in the cards on which they were placed, might make 15; but then there still remained 8 cards over. What was the number of pips in the three first cards?

Ans. 24.

2) From a pack containing a cards, n cards were drawn; upon each of these were so many cards placed, that their number, and the number of pips of the drawn card, were together equal to s. There then remained r cards. What was the number of pips in the first n cards?

Ans. $ns + n + r - a$.

3) A person buys a piece of cloth, and pays $7 for every 5 ells, he then sells every 11 ells for $16, and gains by this bargain $24. How many ells did the piece contain?

Ans. 440.

4) Two travellers began their journeys; A, with $100 in his possession, B with $48. On the road they were attacked by robbers, who took a part of their money from them, by which means A lost twice as much money as B, but still had three times as much left as the latter. How much was taken from each?

Ans. From A $88; from B $44.

5) A book-binder sold me two bound paper books, the one, which contained 48 sheets, for $\frac{7}{12}$ dollar, and the other,

containing 78 sheets, for $\frac{19}{24}$ dollar. The binding and paper cost the same in both. What was the price of the binding?

Ans. 25 cents.

6) A sum of \$156 was to be divided amongst 16 poor children, in proportion to their ages, in such a way, that each of the elder ones received exactly as much more than the next younger, as the next elder received more than he himself. If, therefore, the youngest, according to this division, received \$6, how much more did each of the following ones receive? And how much does the eldest receive?

Ans. \$$\frac{1}{2}$. The eldest received \$13$\frac{1}{2}$.

7) A debt of \$2363, is to be paid off in 34 instalments, so that in each instalment there are \$3 more paid than in the immediately preceding one. How much must be paid in the first instalment?

Ans. \$20.

8) A cistern which is filled by two pipes in 12 minutes, could be filled by one of these pipes in 20 minutes. In what time could it be done by the other pipe?

Ans. In 30 minutes.

9) Required to find a number, such, that when it is added to 15, 27, and 45, there arise three numbers, which are in geometrical progression?

Ans. 9.

10) I have an arithmetical and geometrical progression, each of three terms, and the sum of all six terms = 96. The first term of the arithmetical progression is contained twice in the first term of the geometrical progression; the 2d term of the arithmetical progression is contained thrice in

the 2d term of the geometrical progression, and the 3d term of the arithmetical progression is contained six times in the 3d term of the geometrical progression. What are the progressions?

Ans. 3, 6, 9, and 6, 18, 54.

11) A, B, C, wished to buy a picture, but neither of them had money enough for the purpose. A asked B and C for the half of their money, in order to enable him to buy it; on the other hand, B asked A and C only for the third part of their money; for he then would be enabled to buy it himself. Then C said to A and B, lend me the fourth part of your money and I can buy it. How much money had each, and how much did the picture cost, supposing we knew that the persons had but dollars about them?

Ans. The picture cost either $17, and then A had $5, B $11, and C $13; or the picture cost $34, and then A had $10, B 22, and C $26, &c.

12) Five friends, A, B, C, D, E, jointly spent a certain sum at an inn. This sum is to be paid by one of them; but on counting the dollars they had in their pockets, (for none of them had a smaller coin) they find that none of them had, alone, enough for this purpose. If, then, one is to pay it alone, the others have, for this purpose, to give him yet a part of their money. If A pays, he must receive one fourth; if it is B, he must receive one fifth; if it is C, he must receive one sixth; if D, he must have one seventh; and if E pays, he needs one eighth of the others' money. What did they spend? And how much money had each?

Ans. They spent at least $879; and then A has $319, B $459, C $543, D $599, E $639, or the double, triple, &c, of these numbers.

13) Some persons, who were strangers to each other, agreed to travel in company, sharing the expenses equally, and hired, for this purpose, a coach, for $342. On the road three of them run away; and, on that account, each remaining one had to pay $19 more than otherwise would have fallen to his share. How many persons were there at first?

Ans. 9.

14) An ingot of gold was sold, at a loss, for $420. If it had been sold for $570, then the gain would have been exactly four times as much as the loss is at present. What did it cost?

Ans. $450.

15) Eight oxen have, in seven weeks, eaten all the grass which grew on 400 square-rods of land, in such a manner, that they not only ate all the grass which at first was there, but also that which grew during the time they were grazing. In like manner, have 9 oxen, in eight weeks, ate up all the grass upon 500 square-rods of land. How many oxen can in this way, graze, for 12 weeks, upon 600 square yards of land?

Ans. 8.

16) A man died and left his widow with child, and a clear property of $9000. He ordered in his will, that in case his widow should have a son, he should receive three times as much as his mother; if, however, she had a daughter, then the latter was to receive only half as much as her mother. A short time after the man's death the woman brought forth twins, viz. a son and a daughter. In this case how must the property be divided?

Ans. The mother receives $2000, the son $6000, and the daughter $1000.

17) A traveller starts from a certain place, and makes one mile the first day, two the 2d, three the 3d, four the 4th, and so on, in progression. Five days after, another traveller starts from the same place, takes the same road, and goes 12 miles daily. On what day after the departure of the first, will these three travellers be together?

Ans. On the 8th day, and if they continue their journey at the same rate, they will meet again on the 15th day.

18) The first traveller in the preceding problem goes a miles the first day, but each following day d miles more than he does the preceding one. The 2d starts n days after the first, and goes b miles daily. In what time will they be together?

Ans. After

$$\frac{-(2a-2b-d)\pm\sqrt{[(2a-2b-d)^2-8bdn]}}{2d}\text{ days.}$$

19) Required to find two numbers, such, that their product is equal to their sum, and their sum added to the sum of their squares $= 15\frac{3}{4}$.

Ans. 3 and $\frac{3}{2}$.

20) A shepherd was asked how many sheep he had. He answered, "If I count my sheep by fours, sixes, or nines, I have three over each time; if I count them by sevens or thirteens, I have one over each time; but if I count them by elevens, the remainder is 7." How many sheep had he?

Ans. 183, or 36219, or, &c.

21) Two artillery soldiers, together, filled 1000 cartridges, and used both the same quantity of powder. One says to

the other, "If I had filled as many cartridges as you have, I should have made use of 1800 lbs. of powder." Then the other answers him, "Had I filled as many cartridges as you have, I should have used only 800 lbs. of powder." How many cartridges did each fill? And how much powder did each use?

Ans. The first filled 400, and the 2d 600 cartridges; and each used 1200 lbs. of powder.

22) How long must a dollar remain at 5 per cent compound interest, so that it may become $1000.

Ans. Between 141 and 142 years.

23) A person has a cask of wine, containing 100 gallons, of which each is worth $1\frac{1}{2}$ dollar. From this he draws off 1 gallon of wine, and fills the cask again with water. After the water and wine are well mixed together in the cask, he draws off another gallon, and fills the cask again with water. How often must he repeat to do this, before the mixture in the cask will be worth $1 per gallon?

Ans. Between 40 and 41 times.

24) Divide the number 70 into three such parts, that the first multiplied by 7, the 2d by 8, and the 3d by 9, and then these three products added together, the sum may be = 561. What are the parts?

Ans. 1, 67, 2; or 2, 65, 3; or 3, 63, 4, &c.

25) A schoolmaster gave his pupil two numbers to multiply, of which the one is by 75 greater than the other. When the scholar had finished the multiplication he proved it, by dividing the product by the multiplier (the smaller factor); the quotient was 227, and there remained 113. This proved that it was multiplied wrong, and the schoolmaster ordered

the error to be corrected. When the pupil had found out the error, he said that he had calculated only 1 too little in the multiplication; — "No," said the master, "it was not 1, it was 1000." * What numbers were given to be multiplied?

Ans. 159 and 234.

26) In a solution of salt and water, the weight of the fresh water $= a$, and that of the salt $= b$; therefore its contents $= \frac{b}{a+b}$. How much water must be added to it, that its contents may be $= g$?

Ans. $\frac{b}{g} - (a + b.)$ (The unit of the weight is the same as that in which a and b are given.)

How is this formula to be understood when $\frac{b}{g} < a + b$, consequently also $\frac{b}{a+b} < g$?

27) Any substance A, is p times as heavy as water; another matter B, is only p' times as heavy as water. How much of the 2d matter must be combined with a quantity of the first matter, the weight of which $= g$, so that the mass thus compounded may be p'' times as heavy as water, supposing that p'' falls between p and p'?

Ans. $\frac{gp'(p-p'')}{p(p''-p')}$.

28) Lead is 11,324 times as heavy as water, but the light cork-wood is only 0,24 times, and the heavier fir-wood is but

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* The error being in the fourth place.

0,45 as heavy as water. How much cork must be united to a piece of lead of 60 lbs. so that the composition may weigh exactly as much as a piece of fir of the same size, and consequently may float?

Ans. 65,846 ..... lbs.

29) Let w and x be two quantities, which depend upon one another in such a way, that when the n determinate values, a, a', a'', &c, are assigned to the quantity x, the quantity w, has successively the n corresponding determinate values w, w', w'', &c. How can w be so represented by a polynomial of x, that the given conditions may be fulfilled?

Ans. Let $w = A + Bx + Cx^2 + \ldots\ldots$ &c, and give n terms to the 2d member of this equation, then substitute for w and x the corresponding values, w, a; w', a'; w'', a''; w''', a''', &c; we then obtain n equations, by which the coefficients A, B, C, D, &c, may be found.

30) Let x, y, z, be three quantities, which depend upon each other in such a way, that they simultaneously have the values w, a, b; w', a', b'; w'', a'', b''; &c, respectively. How can w be represented by a polynomial of x and y?

Ans. Let $w = a + Bx + B'y + Cx^2 + C'xy + C''y^2 + Dx^3 + D'x^2y + \ldots\ldots$ &c, and give the 2d member of this equation as many terms as there are corresponding values w, a, b; w' a', b'; &c. Then, substitute for w, x, y, respectively, and successively, the values w, a, b; w', a', b'; w'', a'', b'', &c; and you will obtain exactly as many equations as are necessary to determine the coefficients, A, B, C, D, &c.

31) Let w, x, y, z, be four quantities, which depend upon each other in such a way, that they simultaneously have the values w, a, b, c; w', a', b', c'; w'', a'', b'', c''; &c. How can w be represented by a polynomial of x, y, z?

Ans. Let $w = A + Bx + B'y + B''z + Cx^2 + C'xy + C''xz + C'''y^2 + C''''yz + C'''''z^2 + Dx^3 + D'x^2y + D''x^2z + \ldots\ldots$ &c, and proceed as in the two foregoing problems.

In the same manner we proceed when more quantities, and more corresponding values are given. These problems are, in natural philosophy, as well as in mathematics, of great use. They are also the basis of many analytic methods and formulæ for interpolation.

32) Find a number, such, that if it be multiplied by the square of another number, which is less than itself by 1, the product may $= 1$.

Ans. The number is given by the equation

$$x^3 - 2x^2 + x - 1 = 0.$$

The only real root of this equation is $= 1{,}7548\ldots\ldots$

33) What is the value of the continued fraction

$$\cfrac{1}{q + \cfrac{1}{q + \cfrac{1}{q + \&c}}}$$

ad infinitum, when it has no other quotient than q?

Ans. $\dfrac{-q + \sqrt{(q^2 + 4)}}{2}$.

34) What is the value of the periodical continued fraction

$$\cfrac{1}{q+\cfrac{1}{r+\cfrac{1}{q+\cfrac{1}{r+\&c,}}}}$$

where the quotients q, r, are repeated *ad infinitum*?

Ans. $-\frac{r}{2}+\sqrt{\left(\frac{r^2}{4}+\frac{r}{q}\right)}$.

35) But what value has a continued fraction like the preceding one, when instead of two quotients q, r, three quotients, q, r, s, are repeated *ad infinitum*?

Ans. The value is

$$=-\frac{qrs+q+s-r-\sqrt{[(qrs+q+s+r)^2+4]}}{2(qr+1)}.$$

36) Again, what is the value of such a continued fraction, if the quotients, q, r, s, t, are repeated *ad infinitum*?

Ans. $$-\frac{qrst+qr+qt+st-rs}{2(qrs+q+s)}$$
$$+\frac{\sqrt{[(qrst+qr+qt+st+rs+2)^2-4]}}{2(qrs+q+s)}.$$

What is the law of this value, when there are more than four quotients periodically repeated?

37) In a geometrical proportion the sum of the means $=a$, the sum of the extremes $=b$, the sum of the 4th powers of all four terms $=c$. What is the proportion?

Ans. Let d be equal the difference between the two means, then $d=\sqrt{[-a^2-2b^2\pm 2\sqrt{(c+2a^2b^2)}]}$;

and the proportion required, is

$$\tfrac{1}{2}[b - \sqrt{(b^2 - a^2 + d^2)}] : \tfrac{1}{2}(a - d)$$
$$= \tfrac{1}{2}(a + d) : \tfrac{1}{2}[b + \sqrt{(b^2 - a^2 + d^2)}].$$

38) In a geometrical proportion, the sum of all the terms $= a$, the sum of their squares $= b$, and the sum of their 4th powers $= c$. What is the proportion?

Ans. Let p denote the product of the two extremes, and consequently also of the two means, and d the difference between the sum of the two extremes and the sum of the two means; then $p = \frac{a^2}{2} \pm \sqrt{\frac{a^4 + 2a^2b + b^2 - 2c}{8}}$,

and $d = \sqrt{(2b + 8p - a^2)}$. The four terms of the proportion sought are

$$\tfrac{1}{4}[a - d - \sqrt{[(a - d)^2 - 16p]}$$
$$\tfrac{1}{4}[a + d - \sqrt{[(a + d)^2 - 16p]}$$
$$\tfrac{1}{4}[a + d + \sqrt{[(a + d)^2 - 16p]}$$
$$\tfrac{1}{4}[a - d + \sqrt{[(a - d)^2 - 16p]}.$$

39) A debtor is obliged to pay the sums a, a', a'', a''', &c, in the times n, n', n'', n''', &c; but wishes to pay his whole debt $a + a' + a'' +$ &c, at once. When ought this to take place, if the rate of interest is $= p$, and compound interest is reckoned?

Ans. Let w, w', w'', w''', &c, be the present values of the different payments, so that

$$w = \frac{a}{p^n}, \quad w' = \frac{a'}{p^{n'}}, \quad w'' = \frac{a''}{p^{n''}};$$

then the expression

$$\frac{\log(a + a' + a'' + a''' + \&c) - \log(w + w' + w'' + w''' + \&c)}{\log p}$$

gives the time required.

40) A person owes \$40000. As he is not able to pay them off at once, he agrees with his creditors to pay, beginning from this day, at the end of each year, \$2500, and 5 per cent interest on his debt. When he calculated the time which it would require to discharge the whole of his debt, he found, that at the end of the last year he had not to pay \$2500, but something less. How much less than this sum had he to pay, reckoning compound interest?

Ans. Less than this sum by \$31,88.

41) Required to find four numbers, such, that their sum $= a$, the sum of their squares $= b$, the sum of the twelve products, which arise from multiplying each number by the square of one of the remaining three, $= c$, and the sum of the six products, which arise from multiplying the square of each number by the square of another, $= d$?

Ans. The sum of the products of the four numbers sought, taken two and two, is $= \frac{1}{2}(a^2 - b)$, the sum of their products, taken three and three, $= \frac{1}{6}(a^3 - ab - 2c)$, and the product of all four

$$= \tfrac{1}{24}(a^4 + 2a^2b - 3b^2 - 8ac + 12d).$$

The four numbers, therefore, are given by the following equation:

$$x^4 - ax^3 + \tfrac{1}{2}(a^2 - b)x^2 - \tfrac{1}{6}(a^3 - ab - 2c)x$$
$$+ \tfrac{1}{24}(a^4 + 2a^2b - 3b^2 - 8ac + 12d) = 0.$$

42) Four numbers are given by the following characteristics: Their sum $= a$, the sum of their squares $= b$, the sum of their cubes $= c$, and the sum of their fourth powers $= d$. How are they found?

Ans. The sum of the products of the four numbers sought, taken two and two, $= \frac{1}{2}(a^2 - b)$; the sum of their products, taken three and three, $= \frac{1}{6}(a^3 - 3ab + 2c)$, and the product of all four

$$= \tfrac{1}{24}(a^4 - 6a^2b + 3b^2 + 8ac - 6d).$$

The four numbers, therefore, are given by the following equation:

$$x^4 - ax^3 + \tfrac{1}{2}(a^2 - b)x^2 - \tfrac{1}{6}(a^3 - 3ab + 2c)x$$
$$+ \tfrac{1}{24}(a^4 - 6a^2b + 3b^2 + 8ac - 6d) = 0.$$

THE END.

ERRATA.

Page 2, problem 12, read $5 \times 5 \times 5 \times 5$ instead of $5 \times 5 \times 5$

Page 21, problem 16, read $5\frac{3}{4}$ instead of $3\frac{3}{4}$

Page 22, line 9th, read "smaller" instead of "smallest"

Page 46, problem 22, read $\frac{a^{p+3}b^{m-1}}{12c}$ instead of $\frac{a^{+3}b^{m-1}}{12c}$

Page 74, problem 34, read $3\sqrt{15}$ instead of $2\sqrt{15}$

Page 76, problem 21, read $= \frac{ac}{d}$ instead of $+ \frac{ac}{d}$

Page 77, problem 3, read $(a^n b^p c^q)^{\frac{1}{m}}$ instead of $(a^n b^p c^q)^{\frac{t}{m}}$

Page 91, problem 4, in the denominator, read $8a^{r+5}$ instead of $8a^{+5}$

Page 96, problem 26, read $= 31\sqrt[9]{24}$ instead of $- 31\sqrt[9]{24}$

Page 98, line 11, read "in proportion *to those* of the numbers" instead of "to the numbers"

Page 126, problem 11, 2d line, read $55x^9$ instead of $52x^9$

Page 154, problem 11, read $x - \frac{9x}{2}$ instead of $x = \frac{9x}{2}$

Page 186, at the end of the 1st line, instead of n read $n - m$

Page 201, problem 33, read "$500 more than her children" instead of "$50" &c

Page 203, problem 46, read "each horse-soldier receives *monthly* $5" instead of "$5"

Page 208, problem 65, 4th line, after "*c* units of measure," insert "the 2d over *C*"

Page 220, Ans. to problem 117, read "and" instead of "*apn*"

Page 222, problem 126, 6th line, read "the *whole* capital" instead of "the *same*" &c; and in the Ans., $9\frac{1}{8}$ instead of $9\frac{1}{2}$

Page 223, Ans. to problem 130, read $3\frac{2}{3}$ instead of $5\frac{1}{2}$

Page 226, problem 141, 4th line, read $\$6\frac{1}{6}$ instead of $\$6\frac{1}{2}$

Page 237, problem 19, in the Ans., read $1\frac{23}{27}$ instead of $1\frac{25}{27}$
Page 237, problem 20, read $13\frac{53}{56}$ instead of $13\frac{55}{56}$
Page 271, problem 79, near the end of the 4th line, read $x'^2 x'' + x'x''^2$ instead of $x'^2 x'' + x'x''$
Page 279, problem 30, in the Ans. read $x^3 \mp x^2\sqrt{(2a + b)}$ instead of $x^3 \pm x^2\sqrt{(2a + b)}$
Page 292, problem 40, 2d line of the Ans., in the denominator, read $2\sqrt{a}$ instead of 2
Page 299, problem 7, 2d line, read $12950 instead of $12990
Page 304, line 3d, read $m + 1$ instead of $m + 2$ balls
Page 321, line 1st, read "in" instead of "into"
Page 332, prob. 17, 6th line, read "two" instead of "three"

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